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Hamilton's Principle and the Conservation Theorems of Mathematical Physics

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I. INTRODUCTION

THE derivation of the various conservation theorems arising in the different branches of mathematical physics is usually carried out in each individual instance from a study of the particular equations of motion of the system involved. It has been shown by Klein,[1] Noether,[2] and Bessel-Hagen[3] that when the equations of motion are derivable from a variation principle (Hamilton's principle), a general and systematic procedure for the establishment of the conservation theorems can be developed from a direct study of the variational integral. Since the general equations of mechanics, electromagnetic theory, etc., in use at the present time are derivable from such variational principles, this procedure furnishes the most suitable basis for the systematic study of the conservation theorems.

Despite the fundamental importance of this theory there seems to be no readily available account of it which is adapted to the needs of the student of mathematical physics, while the original papers are not readily accessible.[4,5] It is the object of the present discussion to provide a simplified account of the theory which it is hoped will be of assistance to the reader in gaining an idea of the concepts underlying this important problem. In order to clarify the relationship of the equations of motion and the conservation theorems, as they follow from the variational principle, we shall give a systematic review of the derivations of both sets of equations.

II. THE VARIATIONAL INTEGRAL

The *independent* variables describing the nature of the physical system under discussion will be designated as $x^k (k=1, 2, \cdots, n)$ while the *dependent* variables will be designated as $\psi^\alpha (\alpha=1, 2, \cdots, m)$. The general purpose of the equations of motion is to specify the quantities ψ^α as functions of the independent variables, subject to whatever initial and other boundary conditions may be imposed on the problem. It will be convenient, for the sake of brevity, to refer to the dependent variables as the *state functions* of the system.

The partial derivatives of the state functions with respect to the independent variables will be indicated by the index notation indicated by the following examples,

$$\psi_k{}^\alpha \equiv \partial\psi^\alpha/\partial x^k; \quad \psi_{kl}{}^\alpha \equiv \partial^2\psi^\alpha/\partial x^k \partial x^l. \tag{1}$$

The general assumption underlying Hamilton's principle is that the differential equations of motion are derivable by the application of the variational procedure to an integral of the form,

$$J = \int \mathfrak{L}(x^k, \psi^\alpha, \psi_l{}^\beta) d(x). \tag{2}$$

We shall refer to J as the *variational integral* of the system. The integrand, $\mathfrak{L}$, will be referred to as the *lagrangian density function*, and will be supposed to be a function of the independent variables, and of the state functions and their first partial derivatives, but will be supposed to contain no derivatives of the state functions higher than the first. While this restriction is adequate to cover the cases normally met with in physical problems, the mathematical theory can be generalized to include derivatives of any desired order.[1–3,6] It is supposed that the integral in (2) is to be extended over an arbitrary region of the space of the independent variables, in which $d(x)$ represents the volume element.

III. THE FUNCTIONAL VARIATION OF J

The mathematical problem with which one is concerned is the dependence of the integral, J, on the functional forms of the state functions. We shall start with the consideration of a somewhat more general problem, however, in which we shall allow for the possibility of a change in the region of integration as well as for a change in the functions ψ^α. Specifically, we consider an infinitesimal transformation of the independent variables of the form,

$$x'^k = x^k + \delta x^k, \quad (k=1, 2, \ldots, \eta), \tag{3a}$$

where the quantities δx^k are allowed to be arbitrary infinitesimal functions of the independent variables.[7]

[1] F. Klein, Nachr. kgl. Ges. Wiss. Göttingen, 171 (1918).
[2] E. Noether, Nachr. kgl. Ges. Wiss. Göttingen, 235 (1918).
[3] E. Bessel-Hagen, Math. Ann. **84**, 258 (1921).
[4] A brief discussion is given by R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* (Verlag. Julius Springer, Berlin, 1931), second edition, p. 223.
[5] A treatment of Klein's application of the theory to the conservation theorems of general relativity theory is given by W. Pauli, *Relativitätstheorie* (B. G. Teubner, Leipzig, 1921), Sec. 23.
[6] Th. de Donder, *Théorie Invariantive du Calcul des Variations* (Gauthier-Villars, Paris, 1935), second edition.
[7] The mathematical significance of the quantities, δx^k, $\delta\psi^\alpha$, as arbitrary infinitesimal functions can be made more evident by writing them explicitly in the form,

$$\delta x^k(x) = \lambda\xi^k(x), \quad \delta\psi^\alpha(x) = \lambda\eta^\alpha(x),$$

where $\xi^k(x)$ and $\eta^\alpha(x)$ are arbitrary functions, and λ is to be treated as an infinitesimal parameter of first order. The various orders of infinitesimal terms can be controlled by considering them to be expressed in powers of λ. The reader will find that, after a moderate acquaintance has been obtained with the defini-

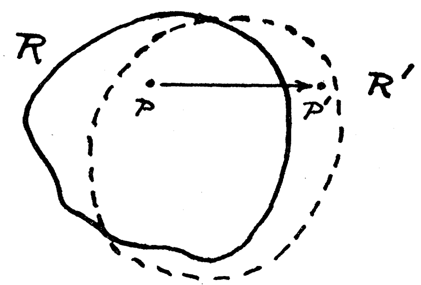

FIG. 1. Schematic diagram of the transformation from region R to region R'.

The geometrical interpretation of Eqs. (3a) is indicated schematically in Fig. 1, in which the original region of integration, R, is mapped to a new region, R', by the point-to-point correspondence indicated in Eqs. (3a).

With Eqs. (3a) we associate also infinitesimal transformations on the state functions, and their partial derivatives, by equations of the form,[7,8]

$$\begin{aligned} \psi'^{\alpha}(x') &= \psi^{\alpha}(x)+\delta\psi^{\alpha}(x), \\ \psi_k{}'^{\alpha}(x') &= \psi_k{}^{\alpha}(x)+\delta\psi_k{}^{\alpha}(x), \end{aligned} \tag{3b}$$

which can be rewritten in the form,

$$\begin{aligned} \delta\psi^{\alpha}(x) &= \psi'^{\alpha}(x')-\psi^{\alpha}(x), \\ \delta\psi_k{}^{\alpha}(x) &= \psi_k{}'^{\alpha}(x')-\psi_k{}^{\alpha}(x). \end{aligned} \tag{3b$'$}$$

The *functional variation* of the integral J is now defined by the relation,

$$\begin{aligned} \delta J &\equiv \int_{R'} \mathfrak{L}(x'^k, \psi'^{\alpha}, \psi_k{}'^{\alpha})d(x') - \int_R \mathfrak{L}(x^k, \psi^{\alpha}, \psi_k{}^{\alpha})d(x) \\ &\equiv \int_{R'} \mathfrak{L}(x^k+\delta x^k, \psi^{\alpha}+\delta\psi^{\alpha}, \psi_k{}^{\alpha}+\delta\psi_k{}^{\alpha})d(x') \\ &\qquad - \int_R \mathfrak{L}(x^k, \psi^{\alpha}, \psi_k{}^{\alpha})d(x). \end{aligned} \tag{4}$$

It is to be noted particularly from this definition that *the functional form of the integrand is not to be altered.* We are seeking to determine the dependence of the integral on the nature of the dependence of $\mathfrak{L}$ on its variables. This point will be of particular importance in our later study of the behavior of J under symmetry transformations of the system (Sec. V).

It is now convenient to reduce the integral over the region R' in Eq. (4) to an integral over the original region R by a change of variables. We have, to first-order quantities,[9]

$$\begin{aligned} \mathfrak{L}(x^k+\delta x^k,\ \psi^{\alpha}+\delta\psi^{\alpha},\ \psi_k{}^{\alpha}+\delta\psi_k{}^{\alpha}) &= \mathfrak{L}(x^k, \psi^{\alpha}, \psi_k{}^{\alpha}) \\ &+(\partial\mathfrak{L}/\partial x^k)\delta x^k+(\partial\mathfrak{L}/\partial\psi^{\alpha})\delta\psi^{\alpha}+(\partial\mathfrak{L}/\partial\psi_k{}^{\alpha})\delta\psi_k{}^{\alpha}. \end{aligned} \tag{5}$$

By the usual convention of a Taylor's series expansion, all of the quantities on the right-hand side of Eq. (5) are supposed to be expressed in terms of the coordinates of the region R.

In Eq. (5) and the later formulas of this paper we shall employ the usual dummy index notation, for both Greek and Latin indices, to indicate summations over the coordinates or state functions, when the corresponding indices are repeated in a given member. Thus, in Eq. (5) the index k in the second and fourth members on the right-hand side is summed from 1 to n, while the index α in the third and fourth members is summed from 1 to m.

The transformation of the volume element from R' to R is accomplished by means of the formula,

$$d(x') = d(x)\cdot\partial(x')/\partial(x), \tag{6}$$

where $\partial(x')/\partial(x)$ is the jacobian of the transformation (3a). A short calculation from Eqs. (3a) shows that to first-order quantities,

$$\partial(x')/\partial(x) = 1+\partial(\delta x^k)/\partial x^k, \tag{7}$$

where the index k on the right-hand side is summed from 1 to n, in accordance with the dummy index notation.

When these results are inserted in Eq. (4), we find that we have, to first-order quantities,

$$\delta J = \int_R \left[\mathfrak{L}\frac{\partial(\delta x^k)}{\partial x^k} + \frac{\partial\mathfrak{L}}{\partial x^k}\delta x^k + \frac{\partial\mathfrak{L}}{\partial\psi^{\alpha}}\delta\psi^{\alpha} + \frac{\partial\mathfrak{L}}{\partial\psi_k{}^{\alpha}}\delta\psi_k{}^{\alpha} \right] d(x). \tag{8}$$

Before the analysis of this formula can be carried further, we must recognize an awkward fact which arises from the definition (3b). Since the functions defined on the two sides of these equations are expressed at different points in the space of the independent variables, it will generally be the case that

$$\delta\psi_k{}^{\alpha} \neq \partial(\delta\psi^{\alpha})/\partial x^k. \tag{9}$$

tions and procedures employed in the calculus of variations, it becomes easy in ordinary cases to keep track of the terms without complicating the notation unduly with such formal considerations; these can always be inserted when needed for clarification.

[8] It is important in definition such as Eq. (3b) to note carefully the variables in terms of which the various functions are expressed, as will appear more definitely in the discussion. On the other hand, it is also important to notice that, as far as the infinitesimal (variational) members are concerned, we can consider them to be expressed in terms of either the variables (x') or (x) as we choose, to first orders of small quantities. This follows from the general formula,

$$\delta\psi^{\alpha}(x') = \delta\psi^{\alpha}(x)+[\partial(\delta\psi^{\alpha})/\partial x^k]\delta x^k+\cdots.$$

Since the second and succeeding members on the right-hand side of this equation are of second and higher order in λ, they can be dropped when we restrict our considerations to first order-terms; and we have to this order

$$\delta\psi^{\alpha}(x') = \delta\psi^{\alpha}(x).$$

We shall make use of this property of the variational terms at a number of points in the discussion for the simplification of the calculations.

[9] The right-hand side of Eq. (5) is to be considered as the expansion, out to first-order terms, in the parameter λ discussed in footnote 7. The fact that the variational functions are expressed in terms of the original coordinates (x) of the region R, to first-order terms, follows from the argument of footnote 8.

It will be convenient to define new quantities, $\delta_*\psi^\alpha$, for which relation (9) will be turned into an equality. To do this we write[10]

$$\left.\begin{aligned}\psi'^\alpha(x')&=\psi^\alpha(x')+\delta_*\psi^\alpha(x'),\\ \psi_k'^\alpha(x')&=\psi_k^\alpha(x')+\delta_*\psi_k^\alpha(x'),\end{aligned}\right\}\tag{10}$$

from which

$$\left.\begin{aligned}\delta_*\psi^\alpha(x')&=\psi'^\alpha(x')-\psi^\alpha(x'),\\ \delta_*\psi_k'^\alpha(x')&=\psi_k'^\alpha(x')-\psi_k^\alpha(x').\end{aligned}\right\}\tag{10$'$}$$

It follows from Eqs. (10) or (10′) that we have

$$\delta_*\psi_k^\alpha(x)=\partial(\delta_*\psi^\alpha)/\partial x^k.\tag{11}$$

Furthermore, by comparison with Eqs. (3a), we see that

$$\left.\begin{aligned}\delta\psi^\alpha&=\delta_*\psi^\alpha+\psi_l^\alpha(x)\cdot\delta x^l,\\ \delta\psi_k^\alpha&=\delta_*\psi_k^\alpha+\psi_{kl}^\alpha(x)\cdot\delta x^l.\end{aligned}\right\}\tag{12}$$

On making use of these relations, we can write Eq. (8) in the form,

$$\delta J=\int_R\bigg[\mathfrak{L}\frac{\partial\delta x^k}{\partial x^k}+\frac{\partial\mathfrak{L}}{\partial x^k}\delta x^k+\frac{\partial\mathfrak{L}}{\partial\psi^\alpha}\delta_*\psi^\alpha +\frac{\partial\mathfrak{L}}{\partial\psi^\alpha}\psi_k^\alpha\delta x^k+\frac{\partial\mathfrak{L}}{\partial\psi_k^\alpha}\delta_*\psi_k^\alpha+\frac{\partial\mathfrak{L}}{\partial\psi_l^\alpha}\psi_{lk}^\alpha\delta x^k\bigg]d(x).\tag{13}$$

In the various differentiations to which we have subjected the function $\mathfrak{L}$ up to this point, we have supposed consistently that, for the purpose of taking partial derivatives, $\mathfrak{L}$ is to be treated explicitly as a function of the independent variables $(x^k, \psi^\alpha, \psi_k^\alpha)$, with k and α running over all appropriate values. It will now be convenient to introduce the concept of partial derivatives with respect to the independent variables when the state functions and their partial derivatives have been supposed to have been substituted as functions of the independent variables. If we use for the symbol of the partial derivatives defined in this sense the notation $\mathfrak{D}/\mathfrak{D}x^k$, inspection shows that, if the function involved contains no partial derivatives of the state function higher than the first, then[11]

$$\mathfrak{D}/\mathfrak{D}x^k\equiv(\partial/\partial x^k)+\psi_k^\alpha(\partial/\partial\psi^\alpha)+\psi_{kl}^\alpha\partial/\partial\psi_l^\alpha.\tag{14}$$

[10] From arguments exactly like those of footnote 8, we can consider the function $\delta_*\psi^\alpha$ and $\delta_*\psi_k^\alpha$ to be expressed in terms of either the variables (x') or (x), to first-order terms. It is convenient in Eqs. (10) and (10′) to express them in terms of the variables (x') in order to establish Eq. (11) with a minimum of discussion. However, in Eq. (11) and Eq. (12) it is more convenient to consider them to be expressed in terms of the original variables (x).

[11] The reader should observe that, despite the somewhat unusual notation, this formula is but a particular case of the ordinary formula of calculus for the implicit differentiation of a function. The use of the notation is justified by the importance of keeping track of the exact definitions being employed. If this formula were to be applied to a function containing higher derivatives than the first of the state functions, further members would need to be added to the right-hand side. For functions of the coordinates alone, only the first member of Eq. (14) would be required. For example, we would have

$$\mathfrak{D}(\delta x^k)/\mathfrak{D}x^l=\partial(\delta x^k)/\partial x^l.$$

The definition given is of convenience because the combination of terms contained on the right-hand side of Eq. (14) appears explicitly in Eq. (13). With this notation we find that Eq. (13) reduces at once to the form,

$$\delta J=\int_R\bigg[\frac{\mathfrak{D}(\mathfrak{L}\delta x^k)}{\mathfrak{D}x^k}+\frac{\partial\mathfrak{L}}{\partial\psi^\alpha}\delta_*\psi^\alpha+\frac{\partial\mathfrak{L}}{\partial\psi_k^\alpha}\delta_*\psi_k^\alpha\bigg]d(x)\tag{15a}$$

$$=\int_R\bigg[\frac{\mathfrak{D}(\mathfrak{L}\delta x^k)}{\mathfrak{D}x_k}+\frac{\partial\mathfrak{L}}{\partial\psi^\alpha}\delta_*\psi^\alpha+\frac{\partial\mathfrak{L}}{\partial\psi_k^\alpha}\frac{\partial(\delta_*\psi^\alpha)}{\partial x^k}\bigg]d(x),\tag{15b}$$

where Eq. (15b) follows directly from Eq. (15a) when we make use of Eq. (11).

We now perform a partial integration of the last member of (15b), employing the formula,

$$\frac{\partial\mathfrak{L}}{\partial\psi_k^\alpha}\frac{\partial(\delta_*\psi^\alpha)}{\partial x^k}=\frac{\mathfrak{D}[(\partial\mathfrak{L}/\partial\psi_k^\alpha)\delta_*\psi^\alpha]}{\mathfrak{D}x^k}-\frac{\mathfrak{D}(\partial\mathfrak{L}/\partial\psi_k^\alpha)}{\mathfrak{D}x^k}\delta_*\psi^\alpha,\tag{16}$$

from which we find that

$$\delta J=\int_R\{(\mathfrak{D}/\mathfrak{D}x^k)[\mathfrak{L}\delta x^k+(\partial\mathfrak{L}/\partial\psi_k^\alpha)\delta_*\psi^\alpha]+[\mathfrak{L}]_\alpha\delta_*\psi^\alpha\}d(x).\tag{17}$$

In this expression we have introduced the quanttiy

$$[\mathfrak{L}]_\alpha\equiv(\partial\mathfrak{L}/\partial\psi^\alpha)-(\mathfrak{D}/\mathfrak{D}x^k)(\partial\mathfrak{L}/\partial\psi_k^\alpha),\tag{18}$$

which is called the *lagrangian derivative* of $\mathfrak{L}$ with respect to ψ^α.

It is customary to express this formula in terms of the variational functions defined in Eqs. (3b), rather than those of Eqs. (10′). On making use of Eqs. (12), we find that

$$\delta J=\int_R\{(\mathfrak{D}/\mathfrak{D}x^k)[(\mathfrak{L}\delta_l^k-(\partial\mathfrak{L}/\partial\psi_k^\alpha)\psi_l^\alpha)\delta x^l+(\partial\mathfrak{L}/\partial\psi_k^\alpha)\delta\psi^\alpha]+[\mathfrak{L}]_\alpha(\delta\psi^\alpha-\psi_m^\alpha\delta x^m)\}d(x),\tag{19}$$

where δ_l^k is the usual Kronecker delta-symbol.

This is our final formula for the functional variation of J. It will scarcely be necessary to emphasize that the discussion is of a purely formal mathematical nature, following from the definition of the functional variation given in Eq. (4), and implies in no way that the results need have any application to any physical system.

IV. THE EQUATIONS OF MOTION (HAMILTON'S PRINCIPLE)

In order to apply the results of Sec. III to a physical system, we assume that the lagrangian density function, $\mathfrak{L}$, describes the properties of the particular system under discussion and that the equations of motion can be derived from Hamilton's principle. According to the assumptions involved in this principle, we restrict ourselves to special types of variations in which (a) the region of integration is unchanged and (b) the variations of the state functions vanish identically over the boundary of the region R. These conditions are given analytically as

$$\delta x^k=0, \quad \delta\psi^\alpha=0 \text{ on the boundary of } R. \tag{20}$$

For variations of this special type it follows at once from Eq. (19) that[11a]

$$\delta J=\int[\mathfrak{L}]_\alpha\cdot\delta\psi^\alpha\cdot d(x). \tag{21}$$

Hamilton's principle now requires that the expression (21) must vanish identically for every choice of the region of integration and for every choice of the variational functions $\delta\psi^\alpha$, subject only to the restriction stated in Eq. (20). It follows from this assumption that

$$[\mathfrak{L}]_\alpha=0 \quad (\alpha=1, 2, \cdots, m). \tag{22}$$

These relations constitute the differential equations of motion of the system involved.

It is of importance for our later discussion to observe explicitly that the choice of the lagrangian density function in the variational integral which will lead to a given set of equations of motion is not unique. The simplest type of change which can be made is the multiplication of $\mathfrak{L}$ by a quantity which does not depend on the variables or state functions of the sytem; that is, the multiplication of $\mathfrak{L}$ by a constant. Since the lagrangian derivative (18) is homogeneous in the function $\mathfrak{L}$, the equations of motion (22) can be multiplied through by any (nonvanishing) constant. This apparently trivial process will be called a *scale transformation.*

Another type of transformation on $\mathfrak{L}$ which leaves the equations of motion invariant is obtained as follows. Let $\Omega^k (k=1, 2, \cdots, n)$ be any set of functions depending on the independent variables and the state functions, but not on the derivatives of the state functions. It will be shown that the two lagrangian density functions $\mathfrak{L}$ and $\mathfrak{L}+\mathfrak{D}\Omega^k/\mathfrak{D}x^k$ both lead to the same set of equations of motion, the proof consisting in the demonstration that under the conditions stated we have identically

$$[\mathfrak{D}\Omega^k/\mathfrak{D}x^k]_\alpha=0. \tag{23}$$

This type of transformation of the density function will be referred to as a *divergence transformation*, since the expression $\mathfrak{D}\Omega^k/\mathfrak{D}x^k$ has the formal appearance of a "divergence" of a generalized vector quantity.[12]

To give the proof of Eq. (23) we note first that from the stated conditions we have

$$\frac{\mathfrak{D}\Omega^k}{\mathfrak{D}x^k}=\frac{\partial\Omega^k}{\partial x^k}+\frac{\partial\Omega^k}{\partial\psi^\beta}\psi_k{}^\beta. \tag{24}$$

It follows by direct calculation from Eq. (24) that

$$\frac{\partial}{\partial\psi^\alpha}\left(\frac{\mathfrak{D}\Omega^k}{\mathfrak{D}x^k}\right)=\frac{\partial}{\partial x^k}\left(\frac{\partial\Omega^k}{\partial\psi^\alpha}\right)+\frac{\partial}{\partial\psi^\beta}\left(\frac{\partial\Omega^k}{\partial\psi^\alpha}\right)\cdot\psi_k{}^\beta=\frac{\mathfrak{D}}{\mathfrak{D}x^k}\left(\frac{\partial\Omega^k}{\partial\psi^\alpha}\right), \tag{25}$$

and that

$$\frac{\partial}{\partial\psi_l{}^\alpha}\left(\frac{\mathfrak{D}\Omega^k}{\mathfrak{D}x^k}\right)=\frac{\partial\Omega^l}{\partial\psi^\alpha}. \tag{26}$$

From Eqs. (25) and (26) we find further that

$$\frac{\mathfrak{D}}{\mathfrak{D}x^l}\left[\frac{\partial}{\partial\psi_l{}^\alpha}\left(\frac{\mathfrak{D}\Omega^k}{\mathfrak{D}x^k}\right)\right]=\frac{\mathfrak{D}}{\mathfrak{D}x^l}\left(\frac{\partial\Omega^l}{\partial\psi^\alpha}\right)=\frac{\partial}{\partial\psi^\alpha}\left(\frac{\mathfrak{D}\Omega^k}{\mathfrak{D}x^k}\right). \tag{27}$$

Assembling these results for the computation of the lagrangian derivative, we obtain

$$\left[\frac{\mathfrak{D}\Omega^k}{\mathfrak{D}x^k}\right]_\alpha\equiv\frac{\partial}{\partial\psi^\alpha}\left(\frac{\mathfrak{D}\Omega^k}{\mathfrak{D}x^k}\right)-\frac{\mathfrak{D}}{\mathfrak{D}x^l}\left[\frac{\partial}{\partial\psi_l{}^\alpha}\left(\frac{\partial\Omega^k}{\partial x^k}\right)\right]$$
$$=\frac{\partial}{\partial\psi^\alpha}\left(\frac{\mathfrak{D}\Omega^k}{\mathfrak{D}x^k}\right)-\frac{\partial}{\partial\psi^\alpha}\left(\frac{\mathfrak{D}\Omega^k}{\mathfrak{D}x^k}\right)=0,$$

which completes the explicit demonstration of Eq. (23).

V. SYMMETRY TRANSFORMATIONS OF A SYSTEM

A transformation of the variables of our system, involving both the independent and the dependent variables, is expressible by the scheme of transformation equations,

$$x'^k=f^k(x), \tag{28a}$$

$$\psi'^\alpha(x')=F^\alpha(\psi(x), x). \tag{28b}$$

The transformations of the derivatives of the state functions are determinable from Eqs. (28b).

In order to maintain the numerical invariance of the variational integral, we allow for a change in the functional form of the lagrangian density function, as specified by the equation,

$$\mathfrak{L}'(x', \psi', \psi_k')d(x')=\mathfrak{L}(x, \psi, \psi_k)d(x). \tag{29}$$

This relation may be regarded as the *definition* of the function, $\mathfrak{L}'$, and justifies the designation of $\mathfrak{L}$ as a *density* (in the space of the independent variables).

In the study of any particular physical system, those transformations which have the property of leaving the equations of motion invariant in form (form-invariant) occupy a place of peculiar significance and are usually designated as the *symmetry transformations* of the system. Symmetry transformations may be of many types.

[11a] On integration of the total derivative member of Eq. (19) one obtains terms which vanish on the boundary of R by relations (20).

[12] In the usual elementary discussions of the invariance of the equations of motion under a divergence transformation, it is not stated whether the functions Ω^k can be permitted to depend explicitly on the state functions and their derivatives. At first glance this would not seem to be of importance, in consideration of the definition of the definition of the operator $\mathfrak{D}/\mathfrak{D}x^k$; but it must be kept in mind that the functional dependence of these functions becomes of importance in the computation of the functional variation of J. The restriction stated in the text will be seen to be of importance in the proof of Eq. (23), for if the Ω^k are allowed to contain the $\psi_k{}^\alpha$ explicitly, then a further member must be added to the right-hand side of Eq. (24) of the form $(\partial\Omega^k/\partial\psi_m{}^\alpha)\psi_{ml}{}^\alpha$, which would introduce *second* partial derivatives of the state functions directly into the variational integral, contrary to the assumption that the density function is to contain no derivatives higher than the first. This restriction can be lifted if one permits the density function to contain derivatives of arbitrary order (references 2, 3, 6).

Their general characteristic is that if a symmetry transformation is applied to any solution of the equations of motion, representing any given type of behavior of the system, it will transform this into another solution representing some other possible motion of the system.[13]

It is clear from these remarks that the symmetry transformations of the system are related intimately to the alteration in the functional form of the lagrangian density function which is implied in Eq. (29). If the equations of motion, expressed in terms of the new variables, are to be of precisely the same functional form as in the old variables, it follows from the discussion of Sec. IV that the two density functions must be related by a divergence transformation.[14] That is, we must have[15]

$$\mathfrak{L}'(x, \psi', \psi_k') = \mathfrak{L}(x', \psi', \psi_k') + \mathfrak{D}\Omega^l/\mathfrak{D}x'^l. \quad (30)$$

Equations (29) and (30) form the basis of the derivation of the conservation theorems associated with symmetry transformations.

The most important type of transformation in the study of conservation theorems is comprised by those which can be developed by the iteration of infinitesimal transformations. In this case it is sufficient to study the infinitesimal transformation itself. We shall use the symbol $\mathfrak{d}$ to represent the infinitesimal changes associated with a symmetry transformation. With a notation corresponding to that of Eqs. (3a, b), the infinitesimal form of Eqs. (28a, b) becomes

$$x'^k = x^k + \mathfrak{d}x^k, \quad \psi'^\alpha(x') = \psi^\alpha(x) + \mathfrak{d}\psi^\alpha(x),$$
$$\psi_k'^\alpha(x') = \psi_k^\alpha(x) + \mathfrak{d}\psi_k^\alpha(x). \quad (31)$$

We shall now use relations (31) to formulate from Eqs. (29) and (30) an equation which can be used as a test of whether or not Eq. (31) represents a symmetry transformation of the system. Equations (29) and (30) now take the forms

$$\mathfrak{L}'(x+\mathfrak{d}x, \psi+\mathfrak{d}\psi, \psi_k+\mathfrak{d}\psi_k)d(x') = \mathfrak{L}(x, \psi, \psi_k)d(x), \quad (32)$$

and

$$\mathfrak{L}'(x+\mathfrak{d}x, \psi+\mathfrak{d}\psi, \psi_k+\mathfrak{d}\psi_k)d(x')$$
$$= \mathfrak{L}(x+\mathfrak{d}x, \psi+\mathfrak{d}\psi, \psi_k+\mathfrak{d}\psi_k)d(x')$$
$$+[\mathfrak{D}(\mathfrak{d}\Omega^k)/\mathfrak{D}x^k]d(x). \quad (33)$$

In the second member on the right-hand side of Eq. (33), we have again taken advantage of the infinitesimal character of the functions $\mathfrak{d}\Omega^k$ to express them in terms of the variables (x) rather than (x').[8] By comparison of Eqs. (32) and (33), we find the relation,

$$\mathfrak{L}(x, \psi, \psi_k)d(x) = \mathfrak{L}(x+\mathfrak{d}x, \psi+\mathfrak{d}\psi, \psi_k+\mathfrak{d}\psi_k)d(x')$$
$$+\mathfrak{D}(\mathfrak{d}\Omega^k)/\mathfrak{D}x^k \cdot d(x). \quad (34)$$

On the rearrangement of Eq. (34) and use of Eq. (7) for the jacobian of the transformation, we find the expression[16]

$$\mathfrak{L}(x+\mathfrak{d}x, \psi+\mathfrak{d}\psi, \psi_k+\mathfrak{d}\psi_k)$$
$$= [\mathfrak{L}(x, \psi, \psi_k) - \mathfrak{D}(\mathfrak{d}\Omega^k)/\mathfrak{D}x^k](1-\partial(\mathfrak{d}x^k)/\partial x^k). \quad (35)$$

We now expand the various members of Eq. (35), retaining only first-order terms, and find the equation,

$$\left[\mathfrak{d}x^k\frac{\partial}{\partial x^k} + \mathfrak{d}\psi^\alpha\frac{\partial}{\partial \psi^\alpha} + \mathfrak{d}\psi_k^\alpha\frac{\partial}{\partial \psi_k^\alpha} + \frac{\partial(\mathfrak{d}x^k)}{\partial x^k}\right]\mathfrak{L} = -\frac{\mathfrak{D}(\mathfrak{d}\Omega^k)}{\mathfrak{D}x^k}. \quad (36)$$

The objective in the application of Eq. (36) to a particular system would usually be to test some supposed symmetry transformation. The test would consist in showing that, on calculation of the left-hand side of Eq. (36) with the assumed density function, the terms could be collected into the form of a divergence expression as given on the right-hand side. In this manner the functions $\mathfrak{d}\Omega^k$ can be identified. In the particularly important case that the left-hand side of Eq. (36) vanishes identically, and the jacobian of the transformation is unity, the density function is said to be *form-invariant* under the transformation. This is the usual case encountered in applications to field theories.

VI. THE CONSERVATION THEOREMS

We shall now make use of Eqs. (29) and (30) for the formulation of the conservation theorem associated with an infinitesimal symmetry transformation. We first rewrite Eq. (34) in the form,

$$\mathfrak{L}(x+\mathfrak{d}x, \psi+\mathfrak{d}\psi, \psi_k+\mathfrak{d}\psi_k)d(x')$$
$$= [\mathfrak{L}(x, \psi, \psi_k) - \mathfrak{D}(\mathfrak{d}\Omega^k)/\mathfrak{D}x^k]d(x). \quad (37)$$

On integration of the right-hand side of Eq. (37) over a region R and the left-hand side over the corresponding region R', we find

$$\int_{R'} \mathfrak{L}(x+\mathfrak{d}x, \psi+\mathfrak{d}\psi, \psi_k+\mathfrak{d}\psi_k)d(x')$$
$$= \int_R [\mathfrak{L}(x, \psi, \psi_k) - \mathfrak{D}(\mathfrak{d}\Omega^k)/\mathfrak{D}x^k]d(x). \quad (38)$$

[13] This general statement of the definition of symmetry transformations would allow the scale and divergence transformations of Sec. IV to be considered as symmetry transformations. However, since they apply to *every* system and depend in no way on the particular functional form of the lagrangian density function, it is not profitable to include them as symmetry transformations.

[14] Our discussion only makes it clear that Eq. (30) is *sufficient* to preserve the form of the equations of motion under the symmetry transformations. The elements of the proof that this condition is also necessary are indicated in Courant-Hilbert (reference 4, p. 165, *et seq.*)

[15] It should be appreciated that we are concerned here only with the study of the functional forms of $\mathfrak{L}$ and $\mathfrak{L}'$, and not primarily with the actual variables in terms of which they are expressed. For the sake of brevity in the succeeding discussion we have expressed them in terms of the primed variables in Eq. (30).

[16] The reader will note that the form of the equation for the jacobian of the transformation which is used in Eq. (35) is

$$\partial(x)/\partial(x') = 1 - \partial(\delta x^k)/\partial x^k,$$

which is obtained as the reciprocal of Eq. (7).

The combination of the two integrals,

$$\mathfrak{d}J=\int_{R'}\mathfrak{L}(x+\mathfrak{d}x,\ \psi+\mathfrak{d}\psi,\ \psi_k+\mathfrak{d}\psi_k)d(x') -\int\mathfrak{L}(x,\ \psi,\ \psi_k)d(x), \quad (39)$$

yields just the functional variation of J under the symmetry transformation (31), according to the fundamental definition of Eq. (4).[17] The insertion of Eq. (39) into Eq. (38) reduces the latter to the form,[18]

$$\mathfrak{d}J+\int_R[\mathfrak{D}(\mathfrak{d}\Omega^k)/\mathfrak{D}x^k]d(x)=0. \quad (40)$$

We can now introduce the actual expression for $\mathfrak{d}J$ from Eq. (19) simply by replacing the variational symbol δ by $\mathfrak{d}$ throughout that formula. This brings Eq. (40) to the form,

$$\int_R\left\{\frac{\mathfrak{D}}{\mathfrak{D}x^k}\left[\left(\mathfrak{L}\delta_l{}^k-\frac{\partial\mathfrak{L}}{\partial\psi_k{}^\alpha}\psi_l{}^\alpha\right)\mathfrak{d}x^l+\frac{\partial\mathfrak{L}}{\partial\psi_k{}^\alpha}\mathfrak{d}\psi^\alpha+\mathfrak{d}\Omega^k\right] +[\mathfrak{L}]_\alpha(\mathfrak{d}\psi^\alpha-\psi_l{}^\alpha\mathfrak{d}x^l)\right\}d(x)=0. \quad (41)$$

Since this integral must vanish identically for every region of integration, R, the integrand must vanish identically, and we have the differential equation,

$$\frac{\mathfrak{D}}{\mathfrak{D}x^k}\left[\left(\mathfrak{L}\delta_l{}^k-\frac{\partial\mathfrak{L}}{\partial\psi_k{}^\alpha}\psi_l{}^\alpha\right)\mathfrak{d}x^l+\frac{\partial\mathfrak{L}}{\partial\psi_k{}^\alpha}\mathfrak{d}\psi^\alpha+\mathfrak{d}\Omega^k\right] +[\mathfrak{L}]_\alpha(\mathfrak{d}\psi^\alpha-\psi_l{}^\alpha\mathfrak{d}x^l)=0. \quad (42)$$

The reader will appreciate that this relation is a direct mathematical consequence of the existence of the symmetry transformation (31), since we have as yet made no use of the equations of motion of the system.

The conservation theorem which we have been seeking now follows at once from Eq. (42) if we impose the condition that the equations of motion of the system are those following from the variational principle, namely, Eqs. (22). This yields the conservation equation,

$$\frac{\mathfrak{D}}{\mathfrak{D}x^k}\left[\left(\mathfrak{L}\delta_l{}^k-\frac{\partial\mathfrak{L}}{\partial\psi_k{}^\alpha}\psi_l{}^\alpha\right)\mathfrak{d}x^l+\frac{\partial\mathfrak{L}}{\partial\psi_k{}^\alpha}\mathfrak{d}\psi^\alpha+\mathfrak{d}\Omega^k\right]=0. \quad (43)$$

It follows from our mode of derivation that each infinitesimal symmetry transformation of the system leads to its own particular conservation relation, and that these conservation equations must be compatible with the lagrangian equations of motion of the system (22). Since the complete set of such symmetry transformations will form a group, we find the conservation theorems associated with this symmetry group of the system.

In the next two sections we present two relatively simple illustrations of the use of the conservation theorems. The examples chosen are the N-body problem of newtonian mechanics and the scalar meson field. The differential equations of the first are the usual equations of particle mechanics, while for the second they are illustrative of the partial differential equations of field theories. The reader will find it instructive to make a comparison of the points of similarity and of divergence which appear in the study of these quite dissimilar cases. The applications to the more complex cases of the electromagnetic field and the fields of particles of higher spin than zero will be left to the reader to develop.

VII. THE CONSERVATION EQUATIONS OF CLASSICAL MECHANICS

The conservation theorems of the N-body problem of newtonian mechanics will be familiar to the reader from his study of elementary mechanics, as will be the equations of motion. We shall therefore devote our attention to the application of the more abstract method which we have developed earlier in this discussion.[3,19]

The lagrangian function of the problem is simply

$$L=T-V=\sum_j\tfrac{1}{2}m_j(\dot{x}_j{}^2+\dot{y}_j{}^2+\dot{z}_j{}^2)-\sum_{i\neq j}Gm_im_j/r_{ij}. \quad (44)$$

There is now but a single independent variable which is the time, t, while the "state functions" are just the coordinates of the particles which we seek to determine as functions of t. The lagrangian function in Eq. (44) is expressed directly in terms of the cartesian coordinates of the particles. In the succeeding discussion we shall also at times designate by the symbol $q^\alpha(\alpha=1, 2, \cdots 3N)$ the whole set of cartesian coordinates of the N particles, arranged in some suitable sequence. When the latter notation is used, the dummy index convention for summations will be retained.

The present formulation of Eq. (36), by means of which we can test the behavior of the lagrangian function, is

$$\left[\mathfrak{d}t\frac{\partial}{\partial t}+\mathfrak{d}q^\alpha\frac{\partial}{\partial q^\alpha}+\mathfrak{d}\dot{q}^\alpha\frac{\partial}{\partial\dot{q}^\alpha}+\frac{d(\mathfrak{d}t)}{dt}\right]L=-\frac{d(\mathfrak{d}\Omega)}{dt}. \quad (45)$$

[17] It is to be emphasized that $\mathfrak{d}J$ is actually the functional derivative of J, as defined in Eq. (4), and is *not* the change in the numerical value of J under Eq. (31), for according to our definition (29), the integral for J is numerically invariant under (33).

[18] Equation (40) can be derived directly from Eq. (36) by comparison with Eq. (8). This means, of course, that Eqs. (36) and (42) are essentially the same; but the procedure adopted in the text appears to be the simplest means of performing the work.

[19] F. Engel, Nachr. kgl. Ges. Wiss. Göttingen, 270 (1916); *ibid.* 189 (1917). F. Engel and K. Faber, *Die Liesche Theorie der Partiellen Differentialgleichungen Erster Ordnung* (B. G. Teubner, Leipzig, 1932).

The appropriate general form of the conservation theorem is found from Eq. (43) to be

$$(d/dt)\{[L-(\partial L/\partial \dot{q}^\alpha)\dot{q}^\alpha]\delta t + (\partial L/\partial \dot{q}^\alpha)\delta q^\alpha + \delta\Omega\} = 0, \quad (46a)$$

or, in the more usual elementary formulation,

$$[L-(\partial L/\partial \dot{q}^\alpha)\dot{q}^\alpha]\delta t + (\partial L/\partial \dot{q}^\alpha)\delta q^\alpha + \delta\Omega = \text{const.} \quad (46b)$$

We can now consider in turn the various special symmetry transformations which are known for this problem, with their appropriate conservation theorems.

(A) Translations of the Axes (Conservation of Momentum)

The simplest symmetry transformations consist of the translations of the origin of the cartesian reference system. Since any such translation can be compounded from translations along the three coordinate directions, it will be sufficient for our purposes to consider only an infinitesimal translation along a single direction, say that of the x axis. The required relations corresponding to Eqs. (33) are

$$t'=t, \quad x'=x-\delta x_0, \quad y'=y, \quad z'=z, \qquad \dot{x}'=\dot{x}, \quad \dot{y}'=\dot{y}, \quad \dot{z}'=\dot{z}. \quad (47)$$

Naturally it is evident from Eq. (44) that L is form-invariant under Eq. (47); and this follows at once formally from Eq. (45), which reduces to the single relation $\Sigma\partial L/\partial x_i=0$ (with $\delta\Omega=0$), which is readily seen to be satisfied from Eq. (44).

Equation (46b) now reduces to the relation,

$$\Sigma_k \partial L/\partial \dot{x}_k = \Sigma_k m_k \dot{x}_k = \text{const}, \quad (48)$$

which expresses the familiar condition of the constancy of the component of the total linear momentum of the system along the x-axis. Taking into account the corresponding theorems for the other two independent directions of translation, we arrive at the vector theorem of conservation of linear momentum.

(B) Translation in the Time (Conservation of Energy)

The infinitesimal transformation in this case is

$$t'=t-\delta t_0, \quad \delta q^\alpha=0, \quad \delta\dot{q}^\alpha=0. \quad (49)$$

We find again that L is form-invariant, condition (45) reducing to the equation $\partial L/\partial t=0$, which is satisfied by Eq. (44).

Equation (46b) now reduces to the equation,

$$L-(\partial L/\partial \dot{q}^\alpha)\dot{q}^\alpha = \text{const}, \quad (50)$$

which is recognizable at once as the theorem of conservation of energy, the function on the left-hand side of Eq. (50) being just the negative of the total energy.

(C) Rotations of the Axes (Conservation of Angular Momentum)

It will be sufficient to consider the rotation of the reference axes about the z axis through an infinitesimal angle $\delta\theta$. The associated transformation equations are

$$t'=t, \quad x'=x+y\cdot\delta\theta, \quad y'=y-x\cdot\delta\theta, \quad z'=z, \qquad \dot{x}'=\dot{x}+\dot{y}\cdot\delta\theta, \quad \dot{y}'=\dot{y}-\dot{x}\cdot\delta\theta, \quad \dot{z}'=\dot{z}. \quad (51)$$

On calculation of the result given by Eq. (45) from Eq. (51), we find

$$\Sigma_i\{[y_i(\partial/\partial x_i)-x_i\partial/\partial y_i]+[\dot{y}_i(\partial/\partial\dot{x}_i)-\dot{x}_i\partial/\partial\dot{y}_i]\}L=0,$$

which vanishes identically from Eq. (44), so that L is again form-invariant, as is evident by inspection, of course.

Equation (46b) now yields the result,

$$\Sigma_i[x_i(\partial L/\partial\dot{y}_i)-y_i\partial L/\partial\dot{x}_i]=\text{const}, \quad (52)$$

which expresses the conservation of the component of angular momentum about the z axis. The vector theorem for the conservation of angular momentum follows by consideration of the rotations about the other two independent space directions.

(D) Velocity Transformations (Center-of-Mass Theorem)

We now examine the conservation theorems associated with the transformations to uniformly moving reference systems. If a motion along the x axis with the infinitesimal velocity δv is taken as typical, the corresponding infinitesimal transformation equations are

$$t'=t, \quad x'=x-t\cdot\delta v, \quad y'=y, \quad z'=z, \qquad \dot{x}'=\dot{x}-\delta v, \quad \dot{y}'=\dot{y}, \quad \dot{z}'=\dot{z}. \quad (53)$$

The left-hand side of Eq. (45) becomes now

$$\delta v\cdot[-t\Sigma_i(\partial L/\partial x_i)-\Sigma_i\partial L/\partial\dot{x}_i].$$

Since $\Sigma_i\partial L/\partial x_i=0$, this reduces at once to

$$-\delta v\cdot\Sigma_i m_i\dot{x}_i = -(d/dt)(\delta v\cdot\Sigma_i m_i x_i).$$

On making comparison with the right-hand side of Eq. (45), we make the identification,

$$\delta\Omega=(\Sigma_i m_i x_i)\cdot\delta v. \quad (54)$$

With this result our conservation theorem (46b) becomes

$$\Sigma_i m_i x_i - t(\Sigma_i m_i\dot{x}_i)=\text{const}. \quad (55)$$

This relation clearly determines the motion of the x component of the center of mass of the system of particles. On combining it with (48), we obtain the result that the center of mass moves with a constant x component of its velocity. With its companion relations referring to motions in the directions of the other two axes, we obtain the vector theorem for the motion of the center of mass. It is therefore quite appropriate that we designate the conservation theorem associated with the transformation to uniformly moving reference systems as the *center-of-mass theorem*.

VIII. THE SCALAR MESON FIELD

This is the simplest example of a field theory and is of current interest in quantum mechanics.[20] The lagrangian density function is

$$\mathfrak{L}=-\tfrac{1}{2}c^2[\eta^{ij}\psi_i\psi_j+\mu^2\psi^2] = -\tfrac{1}{2}c^2[(\text{grad}\psi)^2-(\partial\psi/\partial\tau)^2+\mu^2\psi^2]. \quad (56)$$

The independent variables are $x^0=\tau=ct$, $x^1=x$, $x^2=y$, $x^3=z$, using the ordinary index notation for the space-time variables. The quantities $\eta^{ij}=\eta_{ij}$ are the coefficients in the space-time line element

$$(ds)^2=-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2=\eta_{ij}dx^i dx^j. \quad (57)$$

The coefficient μ in Eq. (56) is expressed in terms of the rest-mass of the meson as $\mu=m_0c/\hbar$.

When the equation of motion (22) is computed from Eq. (56), we find it to be

$$[\nabla^2-(1/c^2)(\partial^2/\partial t^2)-\mu^2]\psi=0. \quad (58)$$

[20] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), Chapter 2.

We note from Eq. (56) that

$$\partial\mathcal{L}/\partial\psi_i=-c^2\eta^{ij}\psi_j,\quad (\partial\mathcal{L}/\partial\psi_i)\psi_i=-c^2\eta^{ij}\psi_i\psi_j,$$

with which the general form of the conservation relation (43) becomes

$$(\mathfrak{D}/\mathfrak{D}x^k)\{[-\tfrac{1}{2}(\eta^{ij}\psi_i\psi_j+\mu^2\psi^2)]\delta x^k +\eta^{kl}\psi_l\psi_m\delta x^m-\eta^{kl}\psi_l\delta\psi\}=0. \quad (59)$$

The symmetry transformations which are involved in this problem are those of the (inhomogeneous) lorentz group, under which the wave function is to be an invariant; i.e., $\psi'(x')=\psi(x)$, so that $\delta\psi=0$. This reduces Eq. (59) to the form,[21]

$$(\mathfrak{D}/\mathfrak{D}x^k)\{-\tfrac{1}{2}(\eta^{ij}\psi_i\psi_j+\mu^2\psi^2)\delta x^k+\eta^{kl}\psi_l\psi_m\delta x^m\}=0. \quad (60)$$

(A) Translations of the Axes (Conservation of Momentum)

The infinitesimal translation along the x axis being given by Eqs. (47), we find the corresponding conservation theorem to be, from Eq. (60),

$$\frac{\partial}{\partial x}\left\{-\frac{1}{2}\left[(\text{grad}\psi)^2-\left(\frac{\partial\psi}{\partial\tau}\right)^2+\mu^2\psi^2\right]+\left(\frac{\partial\psi}{\partial x}\right)^2\right\} +\frac{\partial}{\partial y}\left\{\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x}\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial x}\right\}+\frac{\partial}{\partial\tau}\left\{-\frac{\partial\psi}{\partial\tau}\frac{\partial\psi}{\partial x}\right\}=0. \quad (61)$$

(B) Translation in the Time (Conservation of Energy)

With the transformation (49) the conservation theorem takes the form,

$$\frac{\partial}{\partial x}\left\{-\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial\tau}\right\}+\frac{\partial}{\partial y}\left\{-\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial\tau}\right\}+\frac{\partial}{\partial z}\left\{-\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial\tau}\right\} +\frac{\partial}{\partial\tau}\left\{\frac{1}{2}\left[(\text{grad}\psi)^2+\left(\frac{\partial\psi}{\partial\tau}\right)^2+\mu^2\psi^2\right]\right\}=0. \quad (62)$$

(C) Rotations of the Axes (Conservation of Angular Momentum)

With the transformation (51) for a rotation about the z axis the conservation theorem takes the form,

$$\frac{\partial}{\partial x}\left\{-\frac{1}{2}\left[(\text{grad}\psi)^2-\left(\frac{\partial\psi}{\partial\tau}\right)^2+\mu^2\psi^2\right]y+\frac{\partial\psi}{\partial x}\left(y\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial y}\right)\right\} +\frac{\partial}{\partial y}\left\{+\frac{1}{2}\left[(\text{grad}\psi)^2-\left(\frac{\partial\psi}{\partial\tau}\right)^2+\mu^2\psi^2\right]x+\frac{\partial\psi}{\partial y}\left(y\frac{\partial}{\partial x}-x\frac{\partial\psi}{\partial y}\right)\right\} +\frac{\partial}{\partial z}\left\{\frac{\partial\psi}{\partial z}\left(y\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial y}\right)\right\}+\frac{\partial}{\partial\tau}\left\{-\frac{\partial\psi}{\partial\tau}\left(y\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial y}\right)\right\}=0. \quad (63)$$

(D) Velocity (Lorentz) Transformations (Center-of-Mass Theorem)

In the present case the galilean transformation to moving axes must be replaced by the corresponding lorentz transformation. The substitute for Eqs. (53) for an infinitesimal motion along the x axis with speed δv is

$$x'=x-\tau\cdot\delta\beta,\quad y'=y,\quad z'=z,\quad \tau'=\tau-x\cdot\delta\beta, \quad (64)$$

where $\delta\beta=\delta v/c$. The corresponding conservation theorem is

$$\frac{\partial}{\partial x}\left\{+\frac{1}{2}\left[(\text{grad}\psi)^2-\left(\frac{\partial\psi}{\partial\tau}\right)^2+\mu^2\psi^2\right]\tau+\frac{\partial\psi}{\partial x}\left(-\tau\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial\tau}\right)\right\} +\frac{\partial}{\partial y}\left\{\frac{\partial\psi}{\partial y}\left(-\tau\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial\tau}\right)\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial\psi}{\partial z}\left(-\tau\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial\tau}\right)\right\} +\frac{\partial}{\partial\tau}\left\{+\frac{1}{2}\left[(\text{grad}\psi)^2+\left(\frac{\partial\psi}{\partial\tau}\right)^2+\mu^2\psi^2\right]x-\frac{\partial\psi}{\partial\tau}\left(-\tau\frac{\partial\psi}{\partial x}\right)\right\}=0. \quad (65)$$

From an inspection of the various conservation theorems, we can identify suitable expressions for the dynamical quantities associated with the meson field as follows:

$$\text{energy density}=H=\tfrac{1}{2}[(\text{grad}\psi)^2+(\partial\psi/\partial\tau)^2+\mu^2\psi^2], \quad (66a)$$

$$\text{energy flux vector}=\mathbf{S}=-(\partial\psi/\partial\tau)\,\text{grad}\psi, \quad (66b)$$

$$\text{momentum density}=\mathbf{g}=\mathbf{S}/c^2, \quad (66c)$$

$$\text{angular momentum density}=\mathbf{M}=\mathbf{r}\times\mathbf{g}. \quad (66d)$$

It should be remarked that the definitions (66a, b, c, d) are **not** unique, since we have already observed in Sec. IV that the form of the lagrangian density function can be altered by a divergence transformation without affecting the equations of motion. The problem of establishing a canonical form for the density functions of any given field is usually carried out by demanding, as is done in the theory of relativity, that the equations of conservation of momentum and energy be expressible in the form $\partial T^{ij}/\partial x^j=0$, where T^{ij} is a suitably defined symmetric tensor.[22,23] While such a procedure is important when it is supposed that the dynamical quantities of the field are actually localizable, it has no influence on the definitions of the *total* energy and other entities. If the field is contained within a finite region of space, then by integration of each of the conservation theorems over all space, with discard of those terms which involve the space derivatives, since these lead only to vanishing terms on integration, we obtain the results that the *total* energy, momentum, etc., are independent of time.

Of particular interest is the result obtained from the center-of-mass theorem when the field is contained within a finite region of space, so that the space integrals converge. From Eq. (65) we obtain the result,

$$E\mathbf{R}-\mathbf{P}c^2\cdot\tau=\text{const}, \quad (67)$$

where E and $\mathbf{P}$ are total energy and momentum, defined by the relations,

$$E=\iiint H\,dV,\quad \mathbf{P}=\iiint \mathbf{g}\,dV, \quad (68a)$$

and $\mathbf{R}$ is the "mean center of energy,"

$$\mathbf{R}=\left[\iiint H\mathbf{r}\,dV\right]\Big/E. \quad (68b)$$

The analogy with the corresponding theorem (55) of classical mechanics is obvious.[24]

[21] In the expression of the conservation theorems in explicit form, we have used the notation $\partial/\partial x^k$ instead of $\mathfrak{D}/\mathfrak{D}x^k$ in order to comply with the usual practice.

[22] F. Belinfante, Physica **6**, 887 (1939). L. Rosenfeld, Acad. roy. Belg., classe sci., Mém. **18**, No. 6 (1940).

[23] J. M. Jauch, Acad. Brasileira ciencias **20**, 353 (1948). J. M. Jauch, reference 20, appendix.

[24] For a discussion of the center-of-mass theorem in the theory of relativity the reader is referred to the following papers: M. H. L. Pryce, Proc. Roy. Soc. (London) **A195**, 62 (1948). C. Møller, Communs. Dublin Inst. Advanced Studies, Ser. A, No. 5 (1949).