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A STUDY OF SOME APPROXIMATIONS OF THE PAIRING FORCE

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Abstract: The quasi-particle and the boson approximation are used in the study of the first 0+ excited state of the pairing force: a special case is chosen for its simplicity, which allows us to study quantitatively the validity of these approximations. This validity shows a strong dependence with respect to the coupling constant and to the degeneracy of the shell-model levels.

1. Introduction

The pairing force, which was first introduced by Racah in 1943 as a convenient mathematical tool for the classification of the atomic levels, has been shown in the last few years to be a very good approximation of the short-range part of the nuclear forces for several problems of nuclear physics. Both these reasons, physical interest and mathematical simplicity, led us to study the quasi-particle approximation which is generally used to account for the pairing force [1]), and the boson approximation [2]) which is used for more general problems (especially for the so-called collective vibrations) but which will be shown to be a logical extension of the first one †.

We shall limit ourselves to the study of the first 0+ excited state. The Hamiltonian of a system of nucleons interacting in an average potential by a pairing force is

$$H = \sum_{jm} \varepsilon_j a^\dagger_{jm} a_{jm} + \tfrac{1}{4} G \sum_{\substack{jm \\ j'm'}} (-)^{j-m}(-)^{j'-m'} a^\dagger_{jm} a^\dagger_{j-m} a_{j'-m'} a_{j'm'}, \tag{1}$$

where j labels a shell. We get rid of the magnetic quantum numbers m by defining the following operators:

$$\begin{aligned} A_j{}^\dagger &= \frac{1}{\sqrt{\Omega_j}} \sum_{m>0} (-)^{j-m} a^\dagger_{jm} a^\dagger_{j-m}, \qquad A_j = \frac{1}{\sqrt{\Omega_j}} \sum_{m>0} (-)^{j-m} a_{j-m} a_{jm}, \\ N_j &= \sum_{m} a^\dagger_{jm} a_{jm}, \end{aligned} \tag{2}$$

† See also a similar study by P. W. Anderson [5]), made in the frame of the random-phase approximation in superconductivity.

where Ω_j is the half-degeneracy of the j-shell. These operators obey the following commutation relations:

$$[A_i, A_j^\dagger] = \delta_{ij}\left(1-\frac{N_i}{\Omega_i}\right), \tag{3}$$

$$[N_i, A_j^\dagger] = \delta_{ij} 2A_i^\dagger. \tag{4}$$

Thus one has

$$H = \sum_i \varepsilon_i N_i + G \sum_{ij} \sqrt{\Omega_i}\sqrt{\Omega_j} A_i^\dagger A_j. \tag{1'}$$

Exact solutions are known in two cases:

I. *The degenerate case.* As is well known [3]), the ground state is proportional to $(\sum_i \sqrt{\Omega_i} A_i^\dagger)^{\frac{1}{2}n}|0\rangle$. The first excited 0+ states (seniority 2) are proportional to $(\sum_i \sqrt{\Omega_i} A_i^\dagger)^{\frac{1}{2}n-1} (\sum c_i \sqrt{\Omega_i} A_i^\dagger)|0\rangle$ with $\sum_i c_i \Omega_i = 0$. If p is the number of shells, there are $p-1$ such states, all degenerate, the energy of which is $W = |G|\sum_i \Omega_i$.

II. *The two-particle case.* Looking for an operator $\Gamma^\dagger = \sum_i c_i A_i^\dagger$ such that $H\Gamma^\dagger|0\rangle = W\Gamma^\dagger|0\rangle$, we are led to write

$$[H, \Gamma^\dagger] = \sum_i c_i 2\varepsilon_i A_i^\dagger + G \sum_j \sqrt{\Omega_j} A_j^\dagger \sum_i c_i \sqrt{\Omega_i}\left(1-\frac{N_i}{\Omega_i}\right).$$

The operators N_i acting on the vacuum give no contribution and we find the following eigenvalue equation:

$$\frac{1}{G} = \sum_i \frac{\Omega_i}{W-2\varepsilon_i}, \tag{5}$$

which may be solved graphically. This suggests the following approximation: if we have few particles in rather large shells, we may forget about the Pauli principle, in other words neglect the term N_i/Ω_i in eq. (3). In this approximation, pairs of fermions are just bosons and the Hamiltonian (1') describes a system of bosons in a one-body potential. This problem admits solutions given by eq. (5). However, when we have too many particles, we have first to perform a canonical transformation in order to get a good approximation.

2. The Quasi-Particle and the Quasi-Boson Approximations

2.1 THE QUASI-PARTICLE SCHEME

We define quasi-particles by the Bogoliubov-Valatin transformation

$$\alpha_{jm}^\dagger = u_j a_{jm}^\dagger - v_j (-)^{j-m} a_{j-m}, \qquad u_j^2 + v_j^2 = 1, \tag{6}$$

and, as in the case of particles, we define the following operators:

$$\mathscr{A}_j{}^\dagger = \frac{1}{\sqrt{\Omega_j}} \sum_{m>0} (-)^{j-m} \alpha_{jm}^\dagger \alpha_{j-m}^\dagger, \qquad \mathscr{A}_j = \frac{1}{\sqrt{\Omega_j}} \sum_{m>0} (-)^{j-m} \alpha_{j-m} \alpha_{jm},$$
$$\mathscr{N}_j = \sum_m \alpha_{jm}^\dagger \alpha_{jm}, \tag{7}$$

satisfying the commutator relations

$$[\mathscr{A}_i, \mathscr{A}_j{}^\dagger] = \delta_{ij}\left(1 - \frac{\mathscr{N}_i}{\Omega_i}\right), \tag{8}$$

$$[\mathscr{N}_i, \mathscr{A}_j{}^\dagger] = \delta_{ij} 2\mathscr{A}_i{}^\dagger. \tag{9}$$

We rewrite the pairing Hamiltonian (introducing a chemical potential λ) in terms of these new operators:

$$\begin{aligned}
H &= U + H_{11} + (H_{20} + H_{02}) + H_c + H_{\text{res}}, \\
U &= \sum_i (\varepsilon_i - \lambda) 2 v_i^2 \Omega_i + G(\sum_i \Omega_i u_i v_i)^2, \\
H_{11} &= \sum_i [(\varepsilon_i - \lambda)(u_i^2 - v_i^2) - 2G u_i v_i (\sum_j \Omega_j u_j v_j)] \mathscr{N}_i, \\
H_{20} + H_{02} &= \sum_i [(\varepsilon_i - \lambda) 2 u_i v_i \sqrt{\Omega_i} + G(\sum_j \Omega_j u_j v_j) \sqrt{\Omega_i} (u_i^2 - v_i^2)] (\mathscr{A}_i{}^\dagger + \mathscr{A}_i), \\
H_c &= G \sum_{ij} \sqrt{\Omega_i} \sqrt{\Omega_j} (u_i^2 \mathscr{A}_i{}^\dagger - v_i^2 \mathscr{A}_i)(u_j^2 \mathscr{A}_j - v_j^2 \mathscr{A}_j{}^\dagger), \\
H_{\text{res}} &= G \sum_{ij} \sqrt{\Omega_j} (-u_i v_i) \{\mathscr{N}_i (u_j^2 \mathscr{A}_j - v_j^2 \mathscr{A}_j{}^\dagger) + (u_j^2 \mathscr{A}_j{}^\dagger - v_j^2 \mathscr{A}_j) \mathscr{N}_i\} \\
&\quad + G \sum_{ij} u_i v_i u_j v_j \mathscr{N}_i \mathscr{N}_j.
\end{aligned} \tag{10}$$

The parameters u and v are chosen in order to cancel $H_{20} + H_{02}$. The parameter λ is chosen so that the "vacuum" of the quasi-particle space, defined by $\alpha_i |\phi_0\rangle = 0$, will be an approximation for a system of n particles:

$$\langle \phi_0(\lambda) | N | \phi_0(\lambda) \rangle = n.$$

Consequently, we obtain the well-known equations

$$u_i^2 = \tfrac{1}{2}\left(1 + \frac{\varepsilon_i - \lambda}{E_i}\right), \qquad v_i^2 = \tfrac{1}{2}\left(1 - \frac{\varepsilon_i - \lambda}{E_i}\right), \tag{12}$$

$$\sum_i \frac{\Omega_i}{E_i} = -\frac{2}{G}, \tag{13a}$$

$$\sum_i \Omega_i \left(1 - \frac{\varepsilon_i - \lambda}{E_i}\right) = n \tag{13b}$$

$$E_i = \sqrt{\Delta^2 + (\varepsilon_i - \lambda)^2}, \qquad \Delta = -G \sum_i \Omega_i u_i v_i. \tag{14}$$

Now the Hamiltonian (10) describes a system of quasi-particles interacting in an average potential by means of a two-body force.

The quasi-particle approximation consists in keeping only the free quasi-particle Hamiltonian $H'_0 = U+H_{11}$. The $0+$ excited states are $|\phi_i\rangle = \mathscr{A}_i^\dagger|\phi_0\rangle$, the energy of which is $W_i = 2E_i = 2\sqrt{\Delta^2+(\varepsilon_i-\lambda)^2}$. There are p of them, one more than the right value: this "spurious" state, the existence of which arises from the non-conservation of the number of particles, is mixed up with the others. In the limit where G is very large, we find the degenerate case, and the value $|G|\sum_i \Omega_i$ for the excited states, but we have still a spurious state.

2.2. THE QUASI-BOSON APPROXIMATION

From eq. (10) we see that H_{11} and H_c are of the order of magnitude $G\Omega$ (where Ω is the total degeneracy of the system), while terms of H_{res} are of the order of $G\sqrt{\Omega}$ or even 1 and, besides, include the $\mathscr{N}_i$ operators. If the quasi-particle scheme is a rather good approximation, the low lying states will have only a few quasi-particles. Also if Ω is large, a *consistent* approximation will be to keep only H_c for the interaction of the quasi-particles and neglect the Pauli principle for the pairs of quasi-particles, i.e., use the following approximate commutator rules instead of (8):

$$[\mathscr{A}_i, \mathscr{A}_j^\dagger] = \delta_{ij}. \tag{8'}$$

Now we are led again to a system of bosons with a one-body force. To find the canonical transformation which diagonalizes the Hamiltonian it is convenient to go to the "space" variables. We define

$$q_i = \tfrac{1}{2}\sqrt{2}(\mathscr{A}_i+\mathscr{A}_i^\dagger), \qquad p_i = -\tfrac{1}{2}i\sqrt{2}(\mathscr{A}_i-\mathscr{A}_i^\dagger). \tag{15}$$

Then we get

$$\begin{aligned} H_{11}+H_c &= \tfrac{1}{2}G\sum_i \Omega_i \frac{\varepsilon_i-\lambda}{E_i} + \tilde{H}(q_i, p_i), \\ \tilde{H}(q_i, p_i) &= \sum_i E_i \mathscr{N}_i + \tfrac{1}{2}G\left\{\left(\sum_i \sqrt{\Omega_i}\,\frac{\varepsilon_i-\lambda}{E_i}\,q_i\right)^2 + \left(\sum_i \sqrt{\Omega_i}\,p_i\right)^2\right\}. \end{aligned} \tag{16}$$

We are looking for

$$Q_j = \sum_i \lambda_{ji} q_i, \qquad P_j = \sum_i \mu_{ji} p_i, \tag{17}$$

so that

$$[Q_i, P_j] = i\delta_{ij}, \qquad [Q_i, \tilde{H}] = iB_i P_i, \qquad [P_i, \tilde{H}] = -iC_i Q_i. \tag{18}$$

The energies we are looking for are $W_i = \sqrt{B_i C_i}$.

By straightforward calculations one finds

$$\lambda_{ji} = -\frac{G\sqrt{\Omega_i}}{2E_i}\frac{4E_i^2\Lambda_j^{(1)}+2(\varepsilon_i-\lambda)B_j\Lambda_j^{(2)}}{4E_i^2-W_j^2}$$
$$\mu_{ji} = -G\sqrt{\Omega_i}\frac{C_j\Lambda_j^{(1)}+2(\varepsilon_i-\lambda)\Lambda_j^{(2)}}{4E_i^2-W_j^2}, \tag{19}$$

where $\Lambda_j^{(1)}$ and $\Lambda_j^{(2)}$ are two constants defined by the following equations:

$$\Lambda_j^{(1)} = \sum_i \lambda_{ji}\sqrt{\Omega_i}, \qquad \Lambda_j^{(2)} = \sum_i \mu_{ji}\sqrt{\Omega_i}\,\frac{\varepsilon_i-\lambda}{E_i}. \tag{20}$$

Inserting (19) into (20), we obtain the following equations for $\Lambda_j^{(1)}$ and $\Lambda_j^{(2)}$:

$$\left(1+G\sum_i\frac{\Omega_i 2E_i}{4E_i^2-W_j^2}\right)\Lambda_j^{(1)}+G\left(\sum_i\frac{\Omega_i(\varepsilon_i-\lambda)}{E_i(4E_i^2-W_j^2)}\right)B_j\Lambda_j^{(2)} = 0,$$
$$G\left(\sum_i\Omega_i\frac{\varepsilon_i-\lambda}{E_i(4E_i^2-W_j^2)}\right)C_j\Lambda_j^{(1)}+\left(1+G\sum_i\frac{2(\varepsilon_i-\lambda)^2}{E_i(4E_i^2-W_j^2)}\right)\Lambda_j^{(2)} = 0. \tag{21}$$

Defining

$$a_j = \sum_i\frac{\Omega_i}{2E_i(4E_i^2-W_j^2)}, \qquad b_j = \sum_i\frac{\Omega_i(\varepsilon_i-\lambda)}{E_i(4E_i^2-W_j^2)}, \tag{22}$$

and using (13a), we can write (21) in a simpler way, viz.,

$$W_j^2a_j\Lambda_j^{(1)}+b_jB_j\Lambda_j^{(2)} = 0, \qquad C_jb_j\Lambda_j^{(1)}+(W_j^2-4\Delta^2)a_j\Lambda_j^{(2)} = 0. \tag{21'}$$

The condition of solvability of this equation gives the eigenvalue equation

$$W_j^2(W_j^2-4\Delta^2)a_j^2 = b_j^2W_j^2. \tag{23}$$

There are two cases to be considered.

1) $W_j \neq 0$

Let us put

$$y_j^2 = W_j^2-4\Delta^2. \tag{24}$$

Using (23) and (22) one finds

$$\sum_i\frac{\Omega_i}{E_i(y_j+2(\varepsilon_i-\lambda))} = 0. \tag{25}$$

If Δ and λ are known, these equations can be solved graphically. Let us notice the important fact that there are just $p-1$ solutions. This is the right number of our excited states. We shall see below that the whole spurious state has been taken up by the $W = 0$ solution. The case when the ε_i are symmetrical with respect to the λ-value is of special interest: because of the symmetry, the $y = 0$ solution leads to the lowest eigenvalue $W = 2\Delta$: this is just the gap. From eq. (13b), we see that, in order to obtain this situation, we need two things: a symmetrical distribution of the shell model energy levels, and a number of particles which is just half the total degeneracy of the system †. In the limit where all the levels are degenerate, it is a half-filled shell. In sect. 3, we shall study the case of two levels. One can write (19) in thefollowing form:

$$\lambda_{ji} = \left[\frac{W_j^2\sqrt{\Omega_i}}{E_i(2(\varepsilon_i-\lambda)+y_j)} - y_j\,\frac{\sqrt{\Omega_i}}{E_i}\right]\frac{-GA_j^{(2)}}{2W_j}, \quad \mu_{ji} = \frac{\sqrt{\Omega_i}\,(-GA_j^{(2)})}{2(\varepsilon_i-\lambda)+y_j}; \tag{26}$$

$A_j^{(2)}$ is a normalisation constant so that $\sum_i \lambda_{ji}\mu_{ji} = 1$.

2) $W = 0$

This solution can occur only for $\Delta \neq 0$ and corresponds to $B = 0$. This extra degree of freedom which introduces a continuous spectrum clearly does not belong to our physical problem. In fact, it is easy to see that the wave function $Q_0|\phi_0\rangle$ describes a linear combination of the ground states of the neighbouring nuclei such that the average value of the number of particles is nevertheless equal to n. Eq. s(19) give

$$\lambda_i^0 = \Delta\,\frac{\sqrt{\Omega_i}}{E_i}, \qquad \mu_i^0 = K_0\sqrt{\Omega_i}\,\frac{\Delta^2 a_3+(\varepsilon_i-\lambda)b_3}{E_i^2}, \tag{27}$$

with

$$a_3 = \sum_i \frac{\Omega_i}{E_i^3}, \qquad b_3 = \sum_i \frac{\Omega_i(\varepsilon_i-\lambda)}{E_i^3}, \qquad K_0\Delta(\Delta^2 a_3^2+b_3^2) = 1.$$

The Hamiltonian can be rewritten as follows:

$$\tilde{H} = \tfrac{1}{2}C_0Q_0^2+\tfrac{1}{2}\sum_{j=1}^{p-1} W_j(P_j^2+Q_j^2). \tag{28}$$

The canonical transformation is given by formulas (17), (26), (27). Thus, including H_c in the Hamiltonian leads to the elimination of the spurious state. However, the number of particles is not yet a good quantum number, some terms in H_{res} are playing an important role in this respect.

2.3. THE AVERAGE NUMBER OF PARTICLES IN THE QUASI-BOSON APPROXIMATION

The Bogoliubov-Valatin transformation has been chosen so that the average number of particles in the quasi-particle vacuum has the correct value n:

$$\langle\phi_0|N|\phi_0\rangle = n.$$

† This is precisely the case in superconductivity.

This is not true any more for an excited state. If this excited state is built up with 2 quasi-particles from the j-shell, we have

$$\langle\phi_j|N|\phi_j\rangle = n+2\frac{\varepsilon_j-\lambda}{E_j}.$$

For G large, i.e. Δ large, the extra-term is small. But for $\Delta = 0$, it is just ± 2: the state ϕ_j describes a system with $n+2$ or $n-2$ particles.

In the quasi-boson approximation, let us consider the operators of creation and annihilation of the system of bosons which diagonalize the Hamiltonian

$$\Gamma_j{}^+ = \tfrac{1}{2}\sqrt{2}(Q_j+iP_j), \qquad \Gamma_j = \tfrac{1}{2}\sqrt{2}(Q_j-iP_j),$$

The number of particles in terms of $\mathscr{A}$ and $\mathscr{A}^\dagger$ operators is

$$N = \sum_i \frac{\varepsilon_i-\lambda}{E_i}\mathscr{N}_i+\Delta\sum_i \frac{\sqrt{\Omega_i}}{E_i}(\mathscr{A}_i{}^\dagger+\mathscr{A}_i). \tag{29}$$

The ground state is the vacuum of the new bosons: $\Gamma_j|\psi_0\rangle = 0$. The first excited states are $|\psi_j\rangle = \Gamma_j{}^+|\psi_0\rangle$, and by straightforward calculations (see Appendix 2) one gets

$$\begin{aligned}\langle\psi_0|N-n|\psi_0\rangle &= \sum_{ij}(\mu_{ji}^2+\lambda_{ji}^2)\frac{\varepsilon_i-\lambda}{2E_i}-\sum_i\frac{\varepsilon_i-\lambda}{E_i},\\ \langle\psi_j|N-n|\psi_j\rangle &= \langle\psi_0|N-n|\psi_0\rangle+\sum_i\frac{\varepsilon_i-\lambda}{E_i}(\mu_{ji}^2+\lambda_{ji}^2).\end{aligned} \tag{30}$$

As the $(\varepsilon_i-\lambda)$ have no definite signs, these quantities are rather small in general. One can see that they are identically 0 for the symmetrical case.

If we look at the limit of the degenerate case, we realize that, concerning the average number of particles, the approximation improves as the number of particles increases in the shell, and is best in the middle of the shell.

2.4. THE CASE OF THE COMPLETE SHELLS

As can be seen from eq. (13b), in this case and only in this case, $\Delta = 0$, when $|G|$ is smaller than a critical value $|G_c|$. Then the quasi-particles just define particles and holes, as usually. The eigenvalue equation is

$$\frac{1}{G} = \sum \frac{\Omega_n}{\omega-2(\varepsilon_n-\lambda)}-\frac{\Omega_m}{\omega-2(\varepsilon_m-\lambda)}, \tag{31}$$

where ε_n is above the Fermi surface and ε_m is under the Fermi surface. In this case, one boson creates two particles or two holes, and two bosons are necessary to get the right number of particles. Fig. 1 represents the solution of eq. (31). The energy of the first excited state is indicated by arrows. When $|G|$ increases from 0 to $|G_c|$, the energy of the first excited state decreases, and is 0 when $G = G_c$. Then Δ becomes different from 0 and the quasi-particle scheme starts. This is an illustration of the general relation between the stability of the Hartree Fock solution and the boson approximation (4).

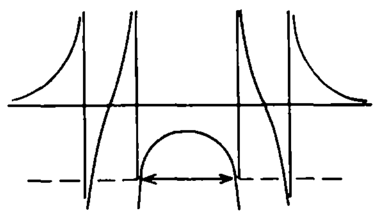

Fig. 1. The solution of eq. (31).

3. The Two-Shell Symmetrical Case

We use this case in order to compare the approximations studied in the last section with the actual solution. Let ε be the distance of the shells, $\pm\frac{1}{2}\varepsilon$ their energy and Ω the common degeneracy; we have $n = 2\Omega$. In the limit of $G = 0$, we have a complete shell, in the limit of $G = -\infty$, a half-filled shell.

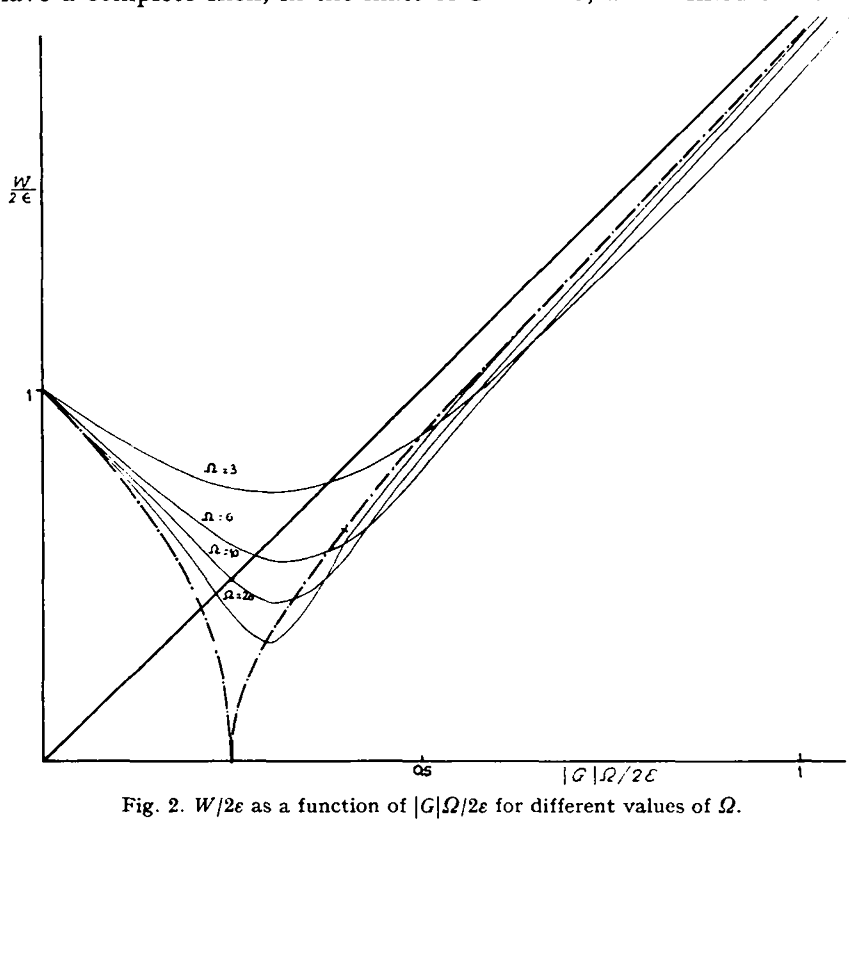

Fig. 2. $W/2\varepsilon$ as a function of $|G|\Omega/2\varepsilon$ for different values of Ω.

The quasi-particle approximation gives 2 degenerate excited states $2E_1 = 2E_2 = 2|G|\Omega$. The critical value G_c, for which $\Delta = 0$, is given by $|G_c|\Omega = \frac{1}{2}\varepsilon$. For $|G| < |G_c|$, the Boson approximation gives $W_b = 2\varepsilon(1-2|G|\Omega/2\varepsilon)^{\frac{1}{2}}$; for $|G| > |G_c|$, the quasi-Boson approximation removes the spurious state. One remains with a one-shell problem, the Hamiltonian of which is

$$H = 2|G|\Omega B^\dagger B - \frac{\varepsilon^2}{8|G|\Omega}(B^\dagger+B)^2, \text{ with } B^\dagger = \tfrac{1}{2}\sqrt{2}(\mathscr{A}_1{}^\dagger - \mathscr{A}_2{}^\dagger). \tag{32}$$

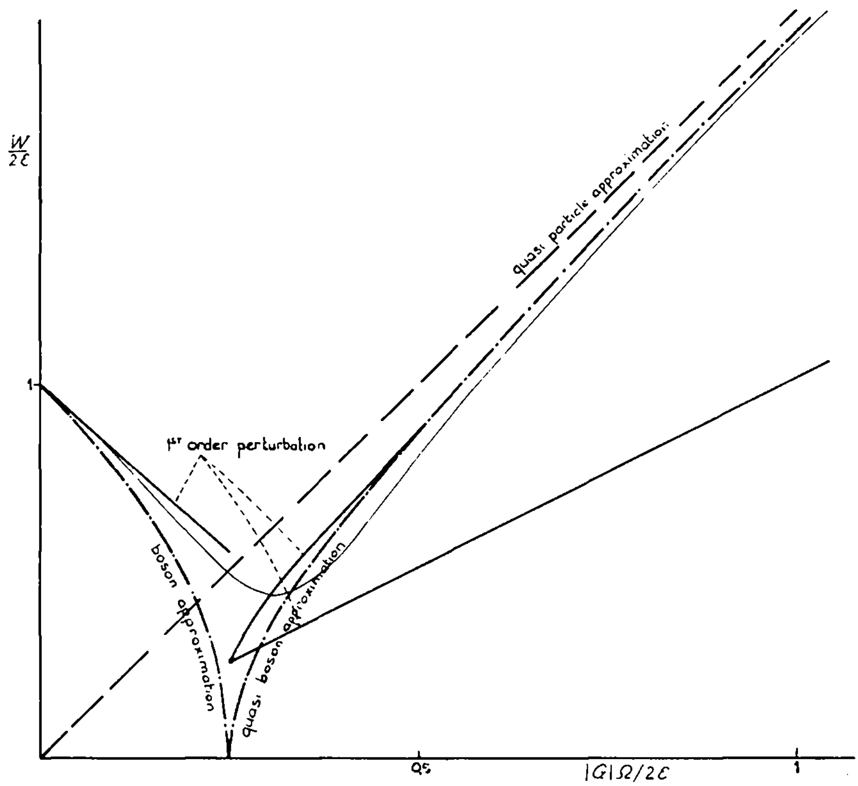

Fig. 3. Comparison of the different approximations (the thin curve is the case $\Omega = 10$ taken from fig. 2).

One clearly sees the interchange of the roles of $|G|\Omega$ and ε as a result of the quasi-particle transformation. Diagonalization leads to the energy $W_{qb} = 2(|G|^2\Omega^2 - \frac{1}{4}\varepsilon^2)^{\frac{1}{2}}$. These curves are plotted in dotted lines on fig. 2. The exact solution was obtained with electronic computors (in Lund and in Paris), for the following values of Ω:

$$\Omega = 3,\ 6,\ 10,\ 20.$$

One can see how the approximation improves when Ω increases. One sees also how good the quasi-boson approximation is for $|G| > |G_c|$. On the other hand, the agreement is not as good for the boson approximation when $|G|<|G_c|$. This might be explained by the fact that, to construct our state, we must here use two bosons instead of one. In fact, a first-order perturbation

$$W_p = 2\varepsilon - 2|G|\Omega\left(1-\frac{1}{\Omega}\right)$$

is even better in this region, as can be seen from fig. 3 (the case $\Omega = 10$). Fig. 3 shows also the result of a first-order perturbation on the free quasi-particle Hamiltonian: the degeneracy is removed, the spurious state is isolated, and the result for the physical state is almost the same as the one obtained by the quasi-boson approximation:

$$W_p = \begin{cases} |G|\Omega, \\ |G|\Omega\left(2-\dfrac{\varepsilon^2}{4|G|^2\Omega^2}\right) & \text{in the limit where } \Omega \text{ is large.} \end{cases}$$

There remains a region around the critical value G_c where no approximation is valid. One could see that, in that region, the average value of the number of quasi-particles in the ground state of the quasi-boson approximation becomes infinite, stressing the fact that the Pauli principle plays a decisive role in this region.

4. Conclusion

We have seen that for the treatment of the pairing force when no exact solution is known, a quasi-particle method with a first order perturbation or a boson approximation gives very satisfactory results, apart for values of the coupling constant in an intermediate region.

The advantage of the boson approximation over the perturbation method is that it removes completely the spurious state of the quasi-particle approximation, and that it can be applied to quite different problems than the one discussed here, for which a perturbation would not give any satisfactory results.

This work was begun during the author's stay at the Institute for Theoretical Physics of Copenhagen. It is a pleasure for her to thank Professors Niels Bohr and Aage Bohr for their kind hospitality, and Professor Mottelson for suggesting this problem, and several illuminating discussions.

Appendix 1.

THE MATRIX ELEMENTS OF THE PAIRING FORCE

We are studying the space generated by A_i operators. An orthonormal set of vectors for this space is

$$|\ldots n_i \ldots\rangle = \prod_i \frac{1}{\sqrt{[n_i]!}} A_i^{\dagger n_i}|0\rangle,$$

with

$$[n_i] = \frac{n_i}{\Omega_i}(\Omega_i - n_i + 1),$$

$$[n_i]! = [n_i][n_i-1]\dots[1] = \frac{1}{\Omega_i^{n_i}} \frac{n_i!\,\Omega_i!}{(\Omega_i-n_i)!}.$$

Then we have

$$A_i^\dagger|\dots n_k \dots\rangle = \sqrt{[n_i+1]} \qquad |\dots n_k + \delta_{ki}\dots\rangle,$$

$$A_i|\dots n_k \dots\rangle = \sqrt{[n_i]} \qquad |\dots n_k - \delta_{ki}\dots\rangle.$$

These equations are very similar to those of real bosons, the difference being in the second term in the definition of $[n_i]$, which comes from the Pauli principle. One finds for the pairing force

$$H|\dots n_k \dots\rangle = (\sum_k 2\varepsilon_k n_k)|\dots n_k\dots\rangle$$
$$+ G\sum_{ij}\sqrt{\Omega_j[n_j-\delta_{ij}+1]\Omega_i[n_i]}|\dots n_k+\delta_{kj}-\delta_{ki}\dots\rangle.$$

Appendix 2.

THE OPERATOR N IN THE QUASI-BOSON APPROXIMATION

We start from the definition of the Q and P operators with respect to the p and q operators:

$$Q = \Lambda q, \qquad P = \mu p, \qquad \text{with } \Lambda\mu^\dagger = 1.$$

We then get the inverse formulas

$$q = \mu^\dagger Q, \qquad p = \Lambda^\dagger P,$$

which lead to the following expression for the $\mathscr{A}$ operators:

$$\mathscr{A} = \tfrac{1}{2}(\mu^\dagger + \Lambda^\dagger)\Gamma + \tfrac{1}{2}(\mu^\dagger - \Lambda^\dagger)\Gamma^\dagger.$$

Thus, we rewrite

$$N = \sum_i \frac{2(\varepsilon_i - \lambda)}{E_i}\mathscr{A}_i^\dagger \mathscr{A}_i + \sqrt{2}\,Q_0 = \mathscr{A}^\dagger \nu \mathscr{A} + \sqrt{2}Q_0$$

as a function of the Γ operators:

$$N = \tfrac{1}{2}\Gamma^\dagger(\mu\nu\mu^\dagger + \Lambda\nu\Lambda^\dagger)\Gamma + \tfrac{1}{4}\Gamma^\dagger(\mu+\Lambda)\nu(\mu^\dagger - \Lambda^\dagger)\Gamma^\dagger$$
$$+\tfrac{1}{4}\Gamma(\mu-\Lambda)\nu(\mu^\dagger+\Lambda^\dagger)\Gamma + \tfrac{1}{4}\mathrm{Tr}(\mu\nu\mu^\dagger + \Lambda\nu\Lambda^\dagger - 2\nu) + \Gamma_0 + \Gamma_0^\dagger.$$

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