Geometry on group manifolds, free motion and QM spectrum

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Notation and definitions:

. Manifold: $\mathcal{M}$, parametrized by coordinates x^μ, $\mu = 1, ...n$.

. Vector field: $\boldsymbol{U} = U^a(x)\boldsymbol{E}_a$ with $U^a(x)$ the components and $\{\boldsymbol{E}_a\}$ the basis of the vector space $T_x\mathcal{M}\,\forall x$. In coordinate basis $\{\boldsymbol{\partial}_\mu\} \rightsquigarrow \boldsymbol{U} = U^\mu(x)\boldsymbol{\partial}_\mu$.
Any basis decomposes as: $\boldsymbol{E}_a = E_a^\mu(x)\boldsymbol{\partial}_\mu$

. 1-form field: linear functional on the space of vectors.
The action on vectors is denoted as $\boldsymbol{\omega}(\boldsymbol{V}) = \langle\boldsymbol{\omega}|\boldsymbol{V}\rangle$.
Given a basis $\{\boldsymbol{E}_a\}$ of V we define $\omega_a(x) \doteq \boldsymbol{\omega}(\boldsymbol{E}_a)$.
$\{\boldsymbol{e}^a\}$ basis of $V^\star$. Dual to $\{\boldsymbol{E}_a\}$ defined as

$$\boldsymbol{e}^a(\boldsymbol{E}_b) = \langle\boldsymbol{e}^a|\boldsymbol{E}_b\rangle = \delta_b^a \quad \Rightarrow \quad \boldsymbol{\omega} = \omega_a(x)\boldsymbol{e}^a$$

Thus, by linearity

$$\boldsymbol{\omega}(\boldsymbol{U}) = \omega_a(x)U^a(x)$$

. Affine connection: asigns to each vector $\boldsymbol{X}$ on $\mathcal{M}$ a differential operator $\nabla_{\boldsymbol{X}}$ which maps arbitrary vectors $\boldsymbol{Y}$ into vectors $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$.

The connection satisfies:
(i) Linearity $\nabla_{f\boldsymbol{X}+g\boldsymbol{Z}}\boldsymbol{Y} = f\nabla_{\boldsymbol{X}}\boldsymbol{Y}+g\nabla_{\boldsymbol{Z}}\boldsymbol{Y}$ and $\nabla_{\boldsymbol{X}}(\boldsymbol{Y}+\boldsymbol{Z}) = \nabla_{\boldsymbol{X}}\boldsymbol{Y}+\nabla_{\boldsymbol{X}}\boldsymbol{Z}$
(ii) $\nabla_{\boldsymbol{X}}f = \boldsymbol{X}(f)$
(iii) Leibniz $\nabla_{\boldsymbol{X}}(f\boldsymbol{Y}) = (\nabla_{\boldsymbol{X}}f)\boldsymbol{Y} + f\,\nabla_{\boldsymbol{X}}\boldsymbol{Y}$.

Linearity implies that knowing its action on a basis $\{\boldsymbol{E}_a\}$ is enough to know its action on any $\boldsymbol{Y}$. Being $\nabla_{\boldsymbol{X}}\boldsymbol{E}_b$ a vector, we can write

$$\nabla_{\boldsymbol{X}}\boldsymbol{E}_b = \boldsymbol{\omega}^a{}_b(\boldsymbol{X})\boldsymbol{E}_a \quad \Rightarrow \quad \nabla_{\boldsymbol{E}_a}\boldsymbol{E}_b = \boldsymbol{\omega}^m{}_b(\boldsymbol{E}_a)\boldsymbol{E}_m = \omega_a{}^m{}_b(x)\,\boldsymbol{E}_m$$

where $\boldsymbol{\omega}^a{}_b = \omega_c{}^a{}_b(x)\,\boldsymbol{e}^c$ are 1-forms. We can generalize the construction stripping away $\boldsymbol{X}$ in (ii) and (iii) to write

$$\nabla f = \boldsymbol{d}f \quad \text{and} \quad \nabla(f\boldsymbol{Y}) = \boldsymbol{d}f \otimes \boldsymbol{Y} + f\,\nabla\boldsymbol{Y}$$

with $\nabla\boldsymbol{Y}$ a $\binom{1}{1}$ tensor. The definition of ∇ on general tensors is obtained by requiring it to satisfy Leibniz on general tensor products

$$\nabla(\boldsymbol{S}\otimes\boldsymbol{T}) = \nabla\boldsymbol{S}\otimes\boldsymbol{T} + \boldsymbol{S}\otimes\nabla\boldsymbol{T}$$

The action on 1-forms follows from Leibniz

$$\nabla_{\boldsymbol{X}}(\boldsymbol{\Omega}(\boldsymbol{Y})) = (\nabla_{\boldsymbol{X}}\boldsymbol{\Omega})(\boldsymbol{Y}) + \boldsymbol{\Omega}(\nabla_{\boldsymbol{X}}\boldsymbol{Y})$$

in terms of a local basis $\{\boldsymbol{E}_a\}$ and $\{\boldsymbol{e}^a\}$ we have

$$\nabla_{\boldsymbol{X}}(\Omega_a Y^a) = (\nabla_{\boldsymbol{X}}\boldsymbol{\Omega})_a Y^a + \Omega_a(\nabla_{\boldsymbol{X}}\boldsymbol{Y})^a$$

since $\nabla_{\boldsymbol{X}}\boldsymbol{Y} = \nabla_{\boldsymbol{X}}(Y^a\boldsymbol{E}_a) = (\boldsymbol{X}(Y^a) + Y^b\boldsymbol{\omega}^a{}_b(\boldsymbol{X}))\boldsymbol{E}_a$, then

$$(\nabla_{\boldsymbol{X}}\boldsymbol{\Omega})_a = \boldsymbol{X}(\Omega_a) - \Omega_b\,\boldsymbol{\omega}^b{}_a(\boldsymbol{X})$$

For $\boldsymbol{\Omega} = \boldsymbol{e}^b$ we conclude that

$$\nabla_{\boldsymbol{E}_a}\boldsymbol{e}^b = -\boldsymbol{\omega}^b{}_m(\boldsymbol{E}_a)\,\boldsymbol{e}^m = -\omega_a{}^b{}_m(x)\,\boldsymbol{e}^m \tag{0.1}$$ `nabe`

PLAYING WITH MATRICES

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Hadamard formula:

$$e^A B e^{-A} = e^{[A,} B \tag{0.2}$$

hada

here the exponential is understood as $e^{[A,} \equiv (1 + [A,\cdot] + \frac{1}{2}[A,[A,\cdot]] + \ldots)$. Then,

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \ldots \tag{0.3}$$

hd

when thinking of this expression in terms of matrix representation of groups, it means that conjugation by a group element is closed on the Lie algebra.

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`Proof`: consider $f(s) = e^{sA} B e^{-sA}$, then

$$\frac{df}{ds} = e^{sA} A B e^{-sA} + e^{sA} B (-A) e^{-sA} = A f(s) - f(s) A = [A, f]$$

From this we find $\ddot{f} = [A, \dot{f}] = [A,[A,f]]$, ... $f^{(n)} = [A,[A..[A,f]]..]$ with n commutators. If we evaluate these expressions at zero and use $f(0) = B$, we obtain $f^{(n)}(0) = [A,[A,..[A,B]]..]$, then

$$f(s) = B + s[A,B] + \frac{s^2}{2}[A,[A,B]] + \frac{s^3}{3!}[A,[A,[A,B]]]....$$

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Duhamel formula: Where do we place the derivative $Z'(t)$ in $e^{Z(t)}$?

$$\frac{d}{dt} e^{Z(t)} = Z' + \frac{1}{2}(Z'\cdot Z + Z\cdot Z') + \frac{1}{3!}(Z'\cdot Z^2 + Z\cdot Z'\cdot Z + Z^2\cdot Z') + ...$$

Everywhere, this is, in all the positions in the expansion! Duhamel formula implements the insertion of Z' in all possible positions of $\exp Z$.

$$\delta e^Z = e^Z \int_0^1 ds\, e^{-sZ} \delta Z e^{sZ} \tag{0.4}$$

duha

Replacing $\delta \to \frac{d}{dt}$ in this formula gives a closed expression for the derivative of e^Z with $Z(t)$ a matrix.

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`Proof`: same trick, take $f(s) = e^{-sZ}\vec{\Delta}(e^{sZ})$ with $\vec{\Delta}$ an operator acting on anything to its right. Then,

$$\frac{df}{ds} = e^{-sZ}(-Z)\vec{\Delta}(e^{sZ}) + e^{-sZ}\vec{\Delta}(Ze^{sZ}) = e^{-sZ}[\vec{\Delta}, Z]e^{sZ}$$

Integrating both sides gives

$$f(1) - f(0) = \int_0^1 ds\, e^{-sZ}[\vec{\Delta}, Z]e^{sZ}$$

The lhs can be worked out to give

$$f(1) - f(0) = e^{-Z}\vec{\Delta}\, e^{Z} - \vec{\Delta} = e^{-Z}(\vec{\Delta}\, e^{Z} - e^{Z}\vec{\Delta}) = e^{-Z}[\vec{\Delta}, e^{Z}]$$

inserting above we find

$$[\vec{\Delta}, e^{Z}] = e^{Z}\int_0^1 ds\, e^{-sZ}[\vec{\Delta}, Z]e^{sZ}$$

Calling $\delta e^{Z} = [\vec{\Delta}, e^{Z}]$ we get (0.4).

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Rewrite the conjugation on the rhs using (0.2) and the definition (0.15)

$$\delta e^{Z} = e^{Z}\int_0^1 ds\, e^{-s\, ad_Z}\delta Z$$

the s-integration on the rhs gives

$$\delta e^{Z} = e^{Z}\left.\frac{e^{-s\, ad_Z}}{-ad_Z}\right|_0^1 \delta Z$$

we conclude that

$$e^{-Z}\delta e^{Z} = \frac{1 - e^{-\, ad_Z}}{ad_Z}\delta Z \qquad (0.5)$$ cl

The rhs should be understood as the expansion $-\sum_{k=0}\frac{(-ad_Z)^k}{(k+1)!}$. The nested commutators show that the left invariant form on the lhs is Lie algebra valued.

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Left invariant forms belong to the Lie algebra: (0.5) can be alternatively obtained in the following way: consider $g(s,t) = e^{sZ(t)}$, then

$$g^{-1}\frac{\partial g}{\partial s} = Z(t)$$

Defining

$$\begin{aligned} B(s,t) &= g^{-1}\frac{\partial g}{\partial t} = e^{-sZ(t)}\frac{\partial e^{sZ(t)}}{\partial t} \\ \Rightarrow \quad \partial_s B &= -ZB + g^{-1}\partial_t(gZ) \\ \Rightarrow \quad \partial_s B &= -ZB + BZ + \dot{Z} \end{aligned} \tag{0.6}$$

we then find that B satisfies

$$\frac{\partial B}{\partial s} = -[Z,B] + \dot{Z} \quad \text{with b.c.} \quad B(0,t) = 0$$

Solving in power series in s

$$\begin{aligned} B(s,t) &= s\dot{Z} + \frac{s^2}{2!}(-ad_Z)\dot{Z} + ... + \frac{s^n}{n!}(-ad_Z)^{n-1}\dot{Z} + ... \\ &= s\,\boldsymbol{\phi}(-s\,ad_Z)\dot{Z} \end{aligned}$$

where $\boldsymbol{\phi}(z) = \frac{e^z - 1}{z}$. Setting $s = 1$ we reobtain (0.5).

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Baker-Campbell-Hausdorff formula: given e^A and e^B it tells us how to write their product as a single exponential

$$e^A e^B = e^Z$$

The result is

$$Z = A + \left(\int_0^1 ds\,\psi(e^{[A,}\ e^{s[B,})\right) B \tag{0.7}$$

bch

with[1]

$$\psi(x) = \frac{x\ln x}{x-1}$$

[1] ψ is the generating function of Bernouilli numbers: $\psi(e^{-y}) = \sum_{n=0}^{\infty} B_n \frac{y^n}{n!}$.

This expression is formal and has a finite radius of convergence. In the case of non-compact Lie groups if A, B are far enough from the identity the series on the rhs diverges. The construction of Z in terms of nested commutators shows, for the case of Lie groups, that Z belongs to the Lie algebra. An explicit expansion with all numerical coefficients was given by Eugene Dynkin in 1947 (see wikipedia).

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`Proof`: consider $e^{Z(s)} = e^A e^{sB}$ then for $\delta = \partial_s$

$$\delta e^{Z(s)} = e^A B e^{sB} = e^Z B \quad \Rightarrow \quad B = e^{-Z}\delta e^Z = \frac{1 - e^{-\,ad_Z}}{ad_Z}\delta Z$$

where we used (0.5). Then

$$\begin{aligned} Z'(s) &= \frac{ad_Z}{1 - e^{-\,ad_Z}} B \\ &= \psi(e^{[Z,}) B \end{aligned} \tag{0.8}$$

Z

where we defined

$$\psi(x) \equiv \frac{x \ln x}{x - 1} = 1 - \sum_{n=1}^{\infty} \frac{(1-x)^n}{n(n+1)}$$

Now, from (0.2)

$$\begin{aligned} e^{[Z,} X &= e^Z X e^{-Z} \\ &= e^A e^{sB} X e^{-sB} e^{-A} \\ &= e^A (e^{s[B,} X) e^{-A} \\ &= e^{[A,} e^{s[B,} X \end{aligned}$$

Inserting this in (0.8) and performing an s-integration one finds

$$Z(1) - Z(0) = \int_0^1 ds\, \psi(e^{[A,} e^{s[B,}) B$$

from $Z(0) = A$ we get (0.7).

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The first few terms of the expansion are

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])-\frac{1}{24}[B,[A,[A,B]]]+\ldots}$$

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Group theory conventions and definitions

IDENTITY - LOCAL

. Group element: $g \in G$. $\forall g$ near the identity we can write[2] $g(t) = \exp(t\boldsymbol{X})$ with $\boldsymbol{X} \in T_eG = Lie(G)$ which we denote $\mathfrak{g}$.

. Group manifold G: we parametrize it with local coordinates ξ^μ, $\mu = 1, ...n$

. Lie algebra generators: $\{\boldsymbol{T}_a\}$ basis of T_eG. Any $\boldsymbol{X} \in \mathfrak{g}$ can be written as

$$\boldsymbol{X} = X^a \boldsymbol{T}_a, \qquad a = 1, \dots n$$

. Structure constants of the Lie algebra $f_a{}^c{}_b$: defined at T_eG. Characterize the group composition law. Writing $g_1(t) = \exp(t\boldsymbol{X})$, $g_2(t) = \exp(t\boldsymbol{Y})$ and $g(t) = \exp(t^2\boldsymbol{Z})$

$$\begin{aligned} g(t) &= g_1(t) g_2(t) g_1^{-1}(t) g_2^{-1}(t) \\ &= 1 + t^2[\boldsymbol{X}, \boldsymbol{Y}] + ... \qquad \rightsquigarrow \quad \boldsymbol{Z} = [\boldsymbol{X}, \boldsymbol{Y}] \end{aligned} \tag{0.9}$$

Linearity of the bracket implies that all information of composition law is contained in

$$[\boldsymbol{T}_a, \boldsymbol{T}_b] = f_a{}^c{}_b \boldsymbol{T}_c$$

Antisymmetry of the commutator implies

$$f_a{}^c{}_b = -f_b{}^c{}_a \tag{0.10}$$

as1

. ad action: action of the Lie algebra on itself. A linear transformation acting on the Lie algebra vector space can be naturally associated to any $\boldsymbol{X} \in \mathfrak{g}$ as

$$\boldsymbol{X} \;\rightarrow\; ad_{\boldsymbol{X}} \boldsymbol{Y} \equiv [\boldsymbol{X}, \boldsymbol{Y}], \quad \forall\, \boldsymbol{Y} \in \mathfrak{g}. \tag{0.11}$$

lad

Hence the map $\boldsymbol{X} \rightarrow ad_{\boldsymbol{X}}$ is a linear representation of the algebra.

[2]All elements in the neighbourhood of the identity can be reached by the exponential map. The vector field could be choosen to be either the Left/Right invariant.

adj action associates a matrix to each Lie algebra basis element $\boldsymbol{T}^a$:

$$ad_{\boldsymbol{T}_a} \to \underbrace{T_a^{(adj)}}_{\text{matrix representation}} : \qquad (T_a^{(adj)})_n^m = f_a{}^m{}_n. \qquad (0.12) \quad \boxed{\texttt{mad}}$$

Jacobi identity obeyed by Lie bracket implies

$$[ad_{\boldsymbol{X}}, ad_{\boldsymbol{Y}}] = ad_{[\boldsymbol{X},\boldsymbol{Y}]} \qquad (0.13) \quad \boxed{\texttt{ad}}$$

. Adjoint action: action of the group on the Lie algebra

$$Ad_g \boldsymbol{Y} \equiv g\boldsymbol{Y}g^{-1}, \quad g \in G, \ \boldsymbol{Y} \in \mathfrak{g} \qquad (0.14) \quad \boxed{\texttt{Adact}}$$

Writing $g(t) = e^{t\boldsymbol{X}}$ one finds using (0.3)

$$Ad_{\exp(t\boldsymbol{X})}\boldsymbol{Y} = e^{t\boldsymbol{X}}\boldsymbol{Y}e^{-t\boldsymbol{X}} = \boldsymbol{Y} + t[\boldsymbol{X},\boldsymbol{Y}] + \frac{1}{2}t^2[\boldsymbol{X},[\boldsymbol{X},\boldsymbol{Y}]] + \ldots$$

The exponential of $ad_{t\boldsymbol{X}}$ action is given by

$$e^{ad_{t\boldsymbol{X}}}\boldsymbol{Y} = \boldsymbol{Y} + t[\boldsymbol{X},\boldsymbol{Y}] + \frac{1}{2}t^2[\boldsymbol{X},[\boldsymbol{X},\boldsymbol{Y}]] + \ldots \qquad (0.15) \quad \boxed{\texttt{aad}}$$

Then,

$$Ad_{\exp(\boldsymbol{X})} = \exp(ad_{\boldsymbol{X}}) \qquad (0.16) \quad \boxed{\texttt{Adad}}$$

. Killing-Cartan form: the matrix representation (0.12) of ad action induces an inner product $\langle\ ,\ \rangle:\ \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle \equiv -\mathsf{tr}[ad_{\boldsymbol{X}} ad_{\boldsymbol{Y}}]. \qquad (0.17) \quad \boxed{\texttt{km}}$$

By linearity, the expansion $\boldsymbol{X} = X^a \boldsymbol{T}_a$ gives

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = X^a Y^b \langle \boldsymbol{T}_a, \boldsymbol{T}_b \rangle$$

reducing the computation of $\langle\ ,\ \rangle$ to the knowledge of

$$\text{Killing-Cartan metric}: \quad \mathfrak{K}_{ab} \equiv \langle \boldsymbol{T}_a, \boldsymbol{T}_b \rangle \quad \Rightarrow \quad \langle \boldsymbol{X}, \boldsymbol{Y} \rangle = X^a Y^b \mathfrak{K}_{ab}$$

$\mathfrak{K}_{ab}$ is called Killing-Cartan metric. It can be written in terms of the structure constants as[3]

$$\mathfrak{K}_{ab} \equiv -\mathsf{tr}[T_a^{(adj)} T_b^{(adj)}] = -f_{a\ \,n}^{\ m} f_{b\ \,m}^{\ n}.$$

The inverse metric $\mathfrak{K}^{ac}$ is defined as usual

$$\mathfrak{K}^{ac}\mathfrak{K}_{cb} = \delta^a_b$$

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`Theorem`: The inner product (0.17) is G-invariant, this means, invariant under the Ad action

$$\langle Ad_g \boldsymbol{Y}, Ad_g \boldsymbol{Z}\rangle = \langle \boldsymbol{Y}, \boldsymbol{Z}\rangle. \qquad (0.18) \quad \boxed{\texttt{adj}}$$

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`Proof`: consider g close to the identity, $g(t) = \exp(t\boldsymbol{X})$ with $t \ll 1$, then using (0.16) we have to first order in t

$$\begin{aligned}\langle \exp(ad_{t\boldsymbol{X}})\boldsymbol{Y}, \exp(ad_{t\boldsymbol{X}})\boldsymbol{Z}\rangle - \langle \boldsymbol{Y}, \boldsymbol{Z}\rangle &= t\,(\langle ad_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}\rangle + \langle \boldsymbol{Y}, ad_{\boldsymbol{X}}\boldsymbol{Z}\rangle) + ... \\ &= t(\langle [\boldsymbol{X}, \boldsymbol{Y}], \boldsymbol{Z}\rangle + \langle \boldsymbol{Y}, [\boldsymbol{X}, \boldsymbol{Z}]\rangle) + ... \\ &= -t\Big(\underbrace{\mathsf{tr}[ad_{[\boldsymbol{X},\boldsymbol{Y}]} ad_{\boldsymbol{Z}}] + \mathsf{tr}[ad_{\boldsymbol{Y}} ad_{[\boldsymbol{X},\boldsymbol{Z}]}]}_{=0}\Big) + ... \\ &= 0 \qquad (0.19) \quad \boxed{\texttt{inv}}\end{aligned}$$

to go to the last equality we used (0.13), cyclicity of trace and Jacobi identity.

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. Totally antisymmetric structure constants: the first line in (0.19) is zero, inserting in it the Lie algebra basis elements $\boldsymbol{T}_a$ one finds

$$\begin{aligned}0 &= \langle ad_{\boldsymbol{T}_a}\boldsymbol{T}_b, \boldsymbol{T}_c\rangle + \langle \boldsymbol{T}_b, ad_{\boldsymbol{T}_a}\boldsymbol{T}_c\rangle \\ &= \langle [\boldsymbol{T}_a, \boldsymbol{T}_b], \boldsymbol{T}_c\rangle + \langle \boldsymbol{T}_b, [\boldsymbol{T}_a, \boldsymbol{T}_c]\rangle \\ &= f_{a\ \,b}^{\ m}\langle \boldsymbol{T}_m, \boldsymbol{T}_c\rangle + f_{a\ \,c}^{\ m}\langle \boldsymbol{T}_b, \boldsymbol{T}_m\rangle \\ &= f_{a\ \,b}^{\ m}\mathfrak{K}_{mc} + f_{a\ \,c}^{\ m}\mathfrak{K}_{bm} \qquad (0.20) \quad \boxed{\texttt{asym}}\end{aligned}$$

[3]For the case of compact semisimple groups, by appropriately normalizing the generators we can set $K_{ab} = \delta_{ab}$.

Defining the lower index structure constants as

$$f_{anb} \equiv \mathfrak{K}_{nm}\, f_{a\ b}^{\ m}$$

eq (0.20) implies

$$f_{acb} + f_{abc} = 0 \quad \Rightarrow \quad f_{abc} = -f_{acb}$$

This relation and (0.10) imply totally antisymmetric structure constants

$$f_{abc} = -f_{cba} = -f_{acb} \tag{0.21}$$ as

In particular (0.21) implies that the adj representation is traceless

$$Tr(T_a^{(adj)}) = f_{a\ m}^{\ m} = \mathfrak{K}^{mn} f_{anm} = 0 \tag{0.22}$$ comp

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GLOBAL - MOVING AROUND

. Maurer-Cartan forms: from the group element matrix representation $g(\xi)$ we construct a Lie algebra valued 1-form field[4]

$$\text{Left invariant form}: \quad \boldsymbol{\sigma}_L \doteq g^{-1}\boldsymbol{d}g = \boldsymbol{e}^a t_a, \qquad \boldsymbol{e}^a = e^a_\mu(\xi)\,\boldsymbol{d}\xi^\mu \tag{0.23}$$ pp

$\{\boldsymbol{e}^a\}$ provides a globally defined basis for $T^\star_g G\ \forall\, g$.
The name left invariant follows from the invariance of $\boldsymbol{e}^a$ under left translations $L:\ G \times G \to G$[5]

$$\text{Left action}: \quad g \to g'(\xi') = g_L(\xi_0)\, g(\xi) \quad \Rightarrow \quad \boldsymbol{e}'^a = \boldsymbol{e}^a \tag{0.24}$$ Lact

[4]The Left/Right globally defined *Maurer-Cartan forms* are linear mapping of the tangent space at each $g \in G$ into the Lie algebra $\mathfrak{g}$, $\boldsymbol{\sigma}:\ T_gG \to T_eG$. The Left invariant MCF is given by the pushforward of a vector in T_gG along the left-translation in the group

$$\boldsymbol{\sigma}_L(\boldsymbol{v}) = (L_{g^{-1}})_*\boldsymbol{v}, \quad \boldsymbol{v} \in T_gG$$

[5]For a given $g \in G,\ L_g:\ G \times G$ where $L_g(h) = gh$. Left/Right actions generate non-linear realizations of the symmetry group.

Right translations induce Ad action on the LIF (0.23)

$$\text{Right action}: \quad g \to g\, g_R^{-1} \;\Rightarrow\; \boldsymbol{e}^a \to g_R\, \boldsymbol{e}^a g_R^{-1} = Ad_{g_R} \boldsymbol{e}^a \tag{0.25}$$ Ract

Right invariant forms $\boldsymbol{f}^a$ are defined in complete analogous fashion

$$\text{Right invariant form}: \quad \boldsymbol{\sigma}_R \doteq \boldsymbol{d}g\, g^{-1} = \boldsymbol{f}^a\, t_a$$

$\{\boldsymbol{f}^a\}$ provide another globally defined basis for $T_g^\star G \;\forall\, g$.

. Maurer-Cartan identities: left invariant forms satisfy

$$\text{LI Maurer-Cartan identity}: \quad \boldsymbol{d}(g^{-1}\boldsymbol{d}g) + g^{-1}\boldsymbol{d}g \wedge g^{-1}\boldsymbol{d}g = 0 \tag{0.26}$$ mc

Calling $\boldsymbol{A} = g^{-1}\boldsymbol{d}g$, this expression reads to the integrability condition

$$\text{Maurer-Cartan} \;\equiv\; \quad \boldsymbol{F}(\boldsymbol{A}) = \boldsymbol{d}\boldsymbol{A} + \boldsymbol{A}^2 = 0$$

showing the existence of a globally defined flat connection over G.
Writing (0.26) using (0.23) we find

$$\begin{aligned} t_a\, \boldsymbol{d}\boldsymbol{e}^a + t_b t_c\, \boldsymbol{e}^b \wedge \boldsymbol{e}^c &= 0 \\ t_a \boldsymbol{d}\boldsymbol{e}^a + \frac{1}{2}[t_b, t_c]\boldsymbol{e}^b \wedge \boldsymbol{e}^c &= 0 \\ t_a \left(\boldsymbol{d}\boldsymbol{e}^a + \frac{1}{2} f_b{}^a{}_c\, \boldsymbol{e}^b \wedge \boldsymbol{e}^c \right) &= 0 \end{aligned} \tag{0.27}$$ spc

Its components in coordinate basis are

$$\partial_\mu e^a_\nu(\xi) - \partial_\nu e^a(\xi)_\mu + f_b{}^a{}_c\, e^b_\mu(\xi) e^c_\nu(\xi) = 0 \tag{0.28}$$ tor

RIF satisfy a Maurer-Cartan identity with a sign shift in the equation

$$\text{RI Maurer-Cartan identity}: \quad \boldsymbol{d}(\boldsymbol{d}g\, g^{-1}) - \boldsymbol{d}g\, g^{-1} \wedge \boldsymbol{d}g\, g^{-1} = 0 \tag{0.29}$$ RMC

. Left invariant vector fields: we define dual vectors $\{\boldsymbol{E}_b\}$ to the $\{\boldsymbol{e}^a\}$ basis in the standard way

$$\boldsymbol{e}^a(\boldsymbol{E}_b) = \langle \boldsymbol{e}^a | \boldsymbol{E}_b \rangle = \delta^a_b, \tag{0.30}$$ dualL

Writing $\boldsymbol{E}_a = E_a^\mu(\xi)\boldsymbol{\partial}_\mu$, we obtain the following relations

$$E_a^\nu(\xi)e_\mu^a(\xi) = \delta_\mu^\nu \quad \text{and} \quad e_\mu^a(\xi)E_b^\mu(\xi) = \delta_b^a \tag{0.31}$$ `vierb`

The set $\{\boldsymbol{E}_b\}$ provides a local basis for $T_gG \; \forall \, g$.

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`Theorem`: structure constants for the $\{\boldsymbol{E}_a\}$ basis are *constant* over the manifold

$$[\boldsymbol{E}_a, \boldsymbol{E}_b] = f_a{}^c{}_b \boldsymbol{E}_c \tag{0.32}$$ `const`

Renaming $\boldsymbol{E}_a \to \boldsymbol{L}_a$ we find left invariant vector fields over the group manifold G. They provide a realization of the Lie algebra as first order differential operators acting on G. The fact of being globally defined make group manifolds parallelizable.

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`Proof`: from (0.31) we get

$$\partial_\nu(e_\mu^a(\xi)E_b^\mu(\xi)) = 0 \quad \Rightarrow \quad \partial_\nu E_b^\rho(\xi) = -E_a^\rho(\xi)\,\partial_\nu e_\mu^a(\xi)\,E_a^\mu(\xi)$$

The commutator (0.32) takes the form

$$\begin{aligned}
[E_a^\mu(\xi)\boldsymbol{\partial}_\mu, E_b^\nu(\xi)\boldsymbol{\partial}_\nu] &= (E_a^\nu(\xi)\partial_\nu E_b^\mu(\xi) - E_b^\nu(\xi)\partial_\nu E_a^\mu(\xi))\boldsymbol{\partial}_\mu \\
&= (E_a^\rho(\xi)E_b^\nu(\xi(\xi)) - E_a^\nu(\xi)E_b^\rho(\xi))\partial_\nu e_\rho^c(\xi)\boldsymbol{E}_c \\
&= E_a^\rho(\xi)E_b^\nu(\xi)(\partial_\nu e_\rho^c(\xi) - \partial_\rho e_\nu^c(\xi))\boldsymbol{E}_c \\
&= -E_a^\rho(\xi)E_b^\nu(\xi)f_m{}^c{}_n e_\nu^m(\xi)e_\rho^n(\xi)\boldsymbol{E}_c \\
&= f_a{}^c{}_b\,\boldsymbol{E}_c
\end{aligned} \tag{0.33}$$ `left`

in going from the third to the fourth line we used (0.28), from the fourth to the last we used (0.31) and the antisymmetry of structure constants.

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. Killing metric over G: with the structure we have we can construct a metric over G as

$$\mathsf{g} = ds^2 = -\mathsf{tr}[g^{-1}\boldsymbol{d}g \otimes g^{-1}\boldsymbol{d}g] \;\Rightarrow\; g_{\mu\nu}(\xi) = e_\mu^a(\xi)e_\nu^b(\xi)\mathfrak{K}_{ab} \tag{0.34}$$ `metr`

For semi-simple compact groups, an appropriate normalization of generators

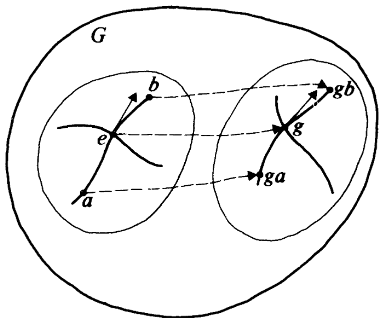

Fig. 1: The left invariant vector fields $\boldsymbol{L}_a$ over G defined as duals to the LIF in (0.30) can be alternatively defined as the pushforward of $\boldsymbol{T}_a$ at the identity by the Left action (0.24). Pfig

puts the KCM in the form[6] $\mathfrak{K}_{ab} = \delta_{ab}$. Computing (0.34) we find

$$\mathsf{g} = \mathfrak{K}_{ab}\, \boldsymbol{e}^a \otimes \boldsymbol{e}^b \tag{0.35}$$

dmetr

The action of g on vector fields $\boldsymbol{U} = U^a(\xi)\boldsymbol{E}_a$ and $\boldsymbol{V} = V^a(\xi)\boldsymbol{E}_a$ is written

$$\begin{aligned}\mathsf{g}(\boldsymbol{U},\boldsymbol{V}) &= \mathfrak{K}_{ab}\,(\boldsymbol{e}^a \otimes \boldsymbol{e}^b)(\boldsymbol{U},\boldsymbol{V}) = \mathfrak{K}_{ab}\boldsymbol{e}^a(\boldsymbol{U})\,\boldsymbol{e}^b(\boldsymbol{V}) \\ &= \mathfrak{K}_{ab}U^a(\xi)V^b(\xi) = g_{\mu\nu}(\xi)U^\mu(\xi)V^\nu(\xi) \end{aligned} \tag{0.36}$$

From (0.34) and (0.31) we obtain the standard relation

$$E^\mu_a(\xi) = \mathfrak{K}_{ab}\, g^{\mu\nu}(\xi) e^b_\nu(\xi) \tag{0.37}$$

vierb2

The KCM can be rephrased as the components of metric in the $\{\boldsymbol{L}_a\}$ basis:

$$\mathsf{g}(\boldsymbol{L}_a, \boldsymbol{L}_b) = \mathfrak{K}_{ab}$$

[6]For non-compact groups the diagonalization brings a Lorentzian like metric, with positive and negative signs.

. Connections on G: any group manifold has a prefered basis given by the LIVF $\boldsymbol{L}_a$[7]. This basis naturally defines a 1-parameter family of connections

$$\nabla^{(\lambda)}_{\boldsymbol{L}_a}\boldsymbol{L}_b = \lambda[\boldsymbol{L}_a,\boldsymbol{L}_b] = \underbrace{\lambda f_a{}^c{}_b}_{\boldsymbol{\omega}^c{}_b(\boldsymbol{L}_a)}\ \boldsymbol{L}_c, \qquad (0.38)$$ Lf

which turns out to be compatible with the Killing metric (0.34)

$$\begin{aligned}\nabla^{(\lambda)}_a \mathsf{g} &= \nabla^{(\lambda)}_a \mathfrak{K}_{mn}\,\boldsymbol{e}^m\otimes\boldsymbol{e}^n + \mathfrak{K}_{mn}\nabla^{(\lambda)}_a\boldsymbol{e}^m\otimes\boldsymbol{e}^n + \mathfrak{K}_{mn}\boldsymbol{e}^m\otimes\nabla^{(\lambda)}_a\boldsymbol{e}^n\\ &= -\mathfrak{K}_{mn}\left(\omega_a{}^m{}_b\,\boldsymbol{e}^b\otimes\boldsymbol{e}^n + \boldsymbol{e}^m\otimes\omega_a{}^n{}_b\,\boldsymbol{e}^b\right)\\ &= -\lambda f_{anb}\left(\boldsymbol{e}^b\otimes\boldsymbol{e}^n + \boldsymbol{e}^n\otimes\boldsymbol{e}^b\right) = 0 \end{aligned} \qquad (0.39)$$

The first term in the first line is zero since $\mathfrak{K}_{ab}$ are constants, in the second line we used (0.1) and the vanishing in the last line follows from the anti-symmetry of the structure constants (see (0.21)).
Among the whole λ-family (0.38), the choice $\lambda = 1/2$ is singled out for being torsion free as follows from (0.27). The choice

$$\boldsymbol{\omega}^a{}_b = \frac{1}{2} f_m{}^a{}_b\,\boldsymbol{e}^m \qquad (0.40)$$ spcn

then gives a metric compatible and torsion free connection over G. Since there is a unique torsion free metric compatible connection, we conclude

$$\nabla^{(1/2)}_a \quad \leftrightarrow \quad \text{Levi-Civita connection for (0.34)}$$

. g bi-invariance and Killing vectors: bi-invariance follows from the invariance of the metric under independent left and right shifts (0.24)-(0.25). Left invariance is immediate, and right invariance follows from (0.18). These imply $G\times G$ isometry group for g.

—

`Theorem`: left and right invariant vector fields are Killing vectors of g closing a $G\times G$ isometry group

$$[\boldsymbol{L}_a,\boldsymbol{L}_b] = f_a{}^c{}_b\boldsymbol{L}_c, \qquad [\boldsymbol{R}_a,\boldsymbol{R}_b] = -f_a{}^c{}_b\boldsymbol{R}_c, \qquad [\boldsymbol{L}_a,\boldsymbol{R}_b] = 0 \qquad (0.41)$$ kil

The first commutator is (0.33), the sign change in the RIVF commutator arises

[7] In fact we have two possible global basis, i.e. $\{\boldsymbol{L}_a\}$ and $\{\boldsymbol{R}_a\}$.

from a sign change in the MCI (0.29), the last follow from the commutative character of Left and Right actions.

—

`Proof`: we need to compute

$$
\begin{aligned}
\pounds_{\boldsymbol{L}_c}\mathtt{g} &= \pounds_{\boldsymbol{L}_c}\big(\mathfrak{K}_{ab}\,\boldsymbol{e}^a\otimes\boldsymbol{e}^b\big)\\
&= \mathfrak{K}_{ab}\big(\pounds_{\boldsymbol{L}_c}(\boldsymbol{e}^a)\otimes\boldsymbol{e}^b+\boldsymbol{e}^a\otimes\pounds_{\boldsymbol{L}_c}(\boldsymbol{e}^b)\big)\\
&= \mathfrak{K}_{ab}\big((i_{\boldsymbol{L}_c}\boldsymbol{d}\boldsymbol{e}^a)\otimes\boldsymbol{e}^b+\boldsymbol{e}^a\otimes(i_{\boldsymbol{L}_c}\boldsymbol{d}\boldsymbol{e}^b)\big)\\
&= -\frac{1}{2}\mathfrak{K}_{ab}\big((i_{\boldsymbol{L}_c}f_b{}^a{}_d\,\boldsymbol{e}^b\wedge\boldsymbol{e}^d)\otimes\boldsymbol{e}^b+\boldsymbol{e}^a\otimes(i_{\boldsymbol{L}_c}f_f{}^b{}_d\,\boldsymbol{e}^f\wedge\boldsymbol{e}^d)\big)\\
&= -\frac{1}{2}\mathfrak{K}_{ab}\big(f_c{}^a{}_d\,\boldsymbol{e}^d\otimes\boldsymbol{e}^b+f_c{}^b{}_d\,\boldsymbol{e}^a\otimes\boldsymbol{e}^d\big)\\
&= -\frac{1}{2}\big(-f_{cdb}\,\boldsymbol{e}^d\otimes\boldsymbol{e}^b+f_{cad}\boldsymbol{e}^a\otimes\boldsymbol{e}^d\big)=0
\end{aligned}
\tag{0.42}
$$

where we used $\pounds_{\boldsymbol{\xi}}\boldsymbol{\omega}=(\boldsymbol{d}i_{\boldsymbol{\xi}}+i_{\boldsymbol{\xi}}\boldsymbol{d})\boldsymbol{\omega}$, the zero torsion condition (0.27), $i_{\boldsymbol{L}_a}\boldsymbol{e}^b=\boldsymbol{e}^b(\boldsymbol{L}_a)=\delta^b_a$ and antisymmetry of the lower index structure constants (0.21).

—

. Riemann curvature tensor: for the Levi-Civita connection (0.40)

$$\begin{aligned} \boldsymbol{R}^a{}_b &\equiv \boldsymbol{d}\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b \\ &= \frac{1}{2} f_c{}^a{}_b \, \boldsymbol{d}\boldsymbol{e}^c + \frac{1}{4} f_m{}^a{}_c \, f_n{}^c{}_b \boldsymbol{e}^m \wedge \boldsymbol{e}^n \\ &= \left(-\frac{1}{4} f_c{}^a{}_b \, f_m{}^c{}_n + \frac{1}{4} f_m{}^a{}_c \, f_n{}^c{}_b \right) \boldsymbol{e}^m \wedge \boldsymbol{e}^n \end{aligned} \tag{0.43}$$

rie

. Ricci tensor:

$$R_{b\nu} \equiv E^\mu_a(\xi) R^a{}_{b\mu\nu}(\xi), \qquad R_{ab} = E^\nu_b(\xi) R_{a\nu}(\xi)$$

from the expression (0.43) one finds

$$R_{ab} = \frac{\mathfrak{K}_{ab}}{4} \quad \text{or} \quad R_{\mu\nu} = \frac{g_{\mu\nu}}{4}$$

Manifesting the fact of the geometry being Einstein and homogeneous.

. Scalar Curvature: being the group a homogeneous space the scalar curvature is constant

$$R \equiv \mathfrak{K}^{ab} R_{ab} = \frac{n}{4}$$

with n the group dimension[8].

[8] One might wonder whether the final result $R = n/4$ depends on the normalization of the generators, the answer is no: any change will scale $\mathfrak{K}$ which compensates upon contracting with its inverse. The 1/4 factor is inherited from the 1/2 in the spin connection.

. Laplace-Beltrami and quadratic Casimirs: we are in position to construct three 2nd order operators. The Laplacian

$$\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu$$

and the quadratic Casimirs for the Left and Right actions, in terms of Killing vectors they are

$$\begin{aligned}\mathcal{C}_L &= \mathfrak{K}^{ab} L_a L_b \\ \mathcal{C}_R &= \mathfrak{K}^{ab} R_a R_b\end{aligned}$$

—

`Theorem`: acting on scalar functions the three operators coincide[9]

$$\Delta = \mathcal{C}_L = \mathcal{C}_R$$

—

`Proof`: we start writing down the Laplace-Beltrami

$$\begin{aligned}\Delta &= \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu) \\ &= g^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu})\partial_\nu \qquad (0.44)\end{aligned}$$ ult

For concreteness we consider the Left Casimir

$$\begin{aligned}\mathcal{C}_L &= \mathfrak{K}^{ab}E_a^\mu\partial_\mu(E_b^\nu\partial_\nu) \\ &= \mathfrak{K}^{ab}E_a^\mu E_b^\nu\partial_\mu\partial_\nu + \mathfrak{K}^{ab}E_a^\mu\partial_\mu(E_b^\nu)\partial_\nu \qquad (0.45)\end{aligned}$$ cass

The first terms in the last lines of (0.44) and (0.45) coincide since

$$g^{\mu\nu}(\xi) = \mathfrak{K}^{ab}E_a^\mu(\xi)E_b^\nu(\xi)$$

So we need to show that the last terms in (0.44) and (0.45) coincide. Calling $e =$

[9]Working with the right invariant vector fields we arrive to the same result

$$\mathcal{C}_R = \mathfrak{K}^{ab}R_aR_b = \Delta$$

$\det e_\mu^a = \exp(\mathsf{tr}\ln e_\mu^a)$, we have $\partial_\mu e = e\, E_a^\nu\, \partial_\mu e_\nu^a$, then

$$\begin{aligned}
\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}) &= \frac{\mathfrak{K}^{ab}}{e}\partial_\mu(eE_a^\mu E_b^\nu) \\
&= \mathfrak{K}^{ab}(E_c^\rho\,\partial_\mu e_\rho^c\, E_a^\mu E_b^\nu - E_c^\mu\,\partial_\mu e_\rho^c\, E_a^\rho E_b^\nu + E_a^\mu \partial_\mu E_b^\nu) \\
&= \mathfrak{K}^{ab}(E_c^\rho(\partial_\mu e_\rho^c - \partial_\rho e_\mu^c + \partial_\rho e_\mu^c)E_a^\mu E_b^\nu - E_c^\mu\,\partial_\mu e_\rho^c\, E_a^\rho E_b^\nu + E_a^\mu \partial_\mu E_b^\nu) \\
&= \mathfrak{K}^{ab}(E_c^\rho(-f_m{}^c{}_n e_\mu^m e_\rho^n) + E_c^\rho\partial_\rho e_\mu^c E_a^\mu E_b^\nu - E_c^\mu\,\partial_\mu e_\rho^c\, E_a^\rho E_b^\nu + E_a^\mu \partial_\mu E_b^\nu) \\
&= \mathfrak{K}^{ab} E_a^\mu \partial_\mu E_b^\nu
\end{aligned}$$

In going from the second to the third line we used the torsion free condition (0.28). The first term in the third line vanishes since it reduces to $f_m{}^c{}_c = 0$ by (0.22) and second and third terms in the same line cancel mutually leading to the answer in the forth line.

—

. Left invariant vector field and right actions: L_a as a differential operator implements the right action

$$e^{\eta^a L_a} g(\xi) = g(\xi)g(\eta) = g(\zeta)$$

At the infinitesimal level $\eta \to 0$, acting on a representation $D^R(g) = e^{X^a T_a^{(R)}}$ we get

$$L_a D(g) = D(g) T_a^{(R)} \qquad (0.46)$$ teren

. Eigenfunctions of the Laplacian on a group manifold:

▷ *The matrix elements of the irreducible representations D^J are eigenfunctions of the Laplacian*

$$\Delta D^J(g(\xi)) = \lambda_J D^J(g(\xi))$$

▷ *The eigenvalue equals the Casimir of the irrep*

$$\lambda_J = \mathcal{C}(J) = \mathfrak{K}^{ab}\, T_a^{(J)} T_b^{(J)}$$

▷ *The $G \times G$ symmetry group of the group manifold is realized on the Laplacian eigenfunctions eigenspace with*

$$\Delta = \mathcal{C}_L = \mathcal{C}_R$$

▷ *The eigenspace degeneracy is $(d_J)^2$ with d_J the dimension of the D^J matrix.*

—

`Proof`: writing $D^J(g(\xi)) = e^{X^a(\xi) T_a^{(J)}}$ for the irrep J

$$\begin{aligned} \Delta D^J(g(\xi)) &= \mathfrak{K}^{ab} L_a L_b D^J(g(\xi)) = \mathfrak{K}^{ab} L_a D^J(g(\xi)) T_b^{(J)} \\ &= \mathfrak{K}^{ab} D^J(g(\xi)) T_a^{(J)} T_b^{(J)} = \lambda_J D^J(g(\xi)) \end{aligned} \tag{0.47}$$

where $\lambda_J = \mathfrak{K}^{ab}\, T_a^{(J)} T_b^{(J)}$ is the quadratic Casimir of the irrep J.

—

Eg: For $G = SU(2)$ with hermitic generators $L_a \to iL_a$ and normalizing $[L_a, L_b] = i\epsilon_{abc} L_c$ we have

$$\vec{L}^2 |j, m, k\rangle = \vec{R}^2 |j, m, k\rangle = j(j+1)|j, m, k\rangle$$

$$L_3 |j, m, k\rangle = m|j, m, k\rangle$$

$$R_3 |j, m, k\rangle = k|j, m, k\rangle$$

The fact of the Casimir having the same value for left and right symmetries arises from bi-invariance. The energy eigenstates are

$$H|j, m, k\rangle = \frac{j(j+1)}{4} |j, m, k\rangle$$

We can figure out the energy level degeneracy immediately because the energy levels only depend on j. There are $2j+1$ possible m values and $2j+1$ possible k values for each value of j, thus the total degeneracy is $(2j+1)^2$.

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