Geometry on group manifolds, free motion and spectrum

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Notation and definitions:

. Vector: $\boldsymbol{U} = U^a \boldsymbol{E}_a$ with U^a the components and $\{\boldsymbol{E}_a\}$ the basis of V the vector space.

Coordinate basis $\boldsymbol{E}_a = \boldsymbol{\partial}_\mu$, then $\boldsymbol{U} = U^\mu \boldsymbol{\partial}_\mu$. Any basis decomposes as: $\boldsymbol{E}_a = E_a^\mu(x)\boldsymbol{\partial}_\mu$

. 1-form: linear functionals on the space of vectors. The action is denoted as $\boldsymbol{\omega}(\boldsymbol{V}) = \langle \boldsymbol{\omega} | \boldsymbol{V} \rangle$.

Given a basis $\{\boldsymbol{E}_a\}$ of V we set $\omega_a = \boldsymbol{\omega}(\boldsymbol{E}_a)$.

$\{\boldsymbol{e}^a\}$ dual or reciprocal basis of $V^\star$ defined as $\boldsymbol{e}^a(\boldsymbol{E}_b) = \langle \boldsymbol{e}^a | \boldsymbol{E}_b \rangle = \delta^a_b$ then $\boldsymbol{\omega} = \omega_a \boldsymbol{e}^a$.

Thus

$$\boldsymbol{\omega}(\boldsymbol{U}) = \omega_a U^a$$

. Affine connection: asigns to each vector $\boldsymbol{X}$ on $\mathcal{M}$ a differential operator $\nabla_{\boldsymbol{X}}$ which maps arbitrary vectors $\boldsymbol{Y}$ into vectors $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$. The connection satisfies:

(i) linearity $\nabla_{f\boldsymbol{X}+g\boldsymbol{Z}}\boldsymbol{Y} = f\nabla_{\boldsymbol{X}}\boldsymbol{Y} + g\nabla_{\boldsymbol{Z}}\boldsymbol{Y}$ and $\nabla_{\boldsymbol{X}}(\boldsymbol{Y}+\boldsymbol{Z}) = \nabla_{\boldsymbol{X}}\boldsymbol{Y} + \nabla_{\boldsymbol{X}}\boldsymbol{Z}$

(ii) $\nabla_{\boldsymbol{X}} f = \boldsymbol{X}(f)$

(iii) Leibniz $\nabla_{\boldsymbol{X}}(f\boldsymbol{Y}) = (\nabla_{\boldsymbol{X}} f)\boldsymbol{Y} + f\,\nabla_{\boldsymbol{X}}\boldsymbol{Y}$.

It suffices to see it work in a basis $\{\boldsymbol{E}_a\}$, being a vector we expect the decomposition

$$\nabla_{\boldsymbol{X}} \boldsymbol{E}_b = \boldsymbol{\omega}^a{}_b(\boldsymbol{X})\boldsymbol{E}_a \quad \Rightarrow \quad \nabla_{\boldsymbol{E}_a} \boldsymbol{E}_b = \boldsymbol{\omega}^m{}_b(\boldsymbol{E}_a)\boldsymbol{E}_m = \omega_a{}^m{}_b\, \boldsymbol{E}_m$$

where $\boldsymbol{\omega}^a{}_b = \omega_c{}^a{}_b\, \boldsymbol{e}^c$ are 1-forms. We can generalize the construction stripping away $\boldsymbol{X}$ in (ii) and (iii) to write

$$\nabla f = \boldsymbol{d}f \quad \text{and} \quad \nabla(f\boldsymbol{Y}) = \boldsymbol{d}f \otimes \boldsymbol{Y} + f\, \nabla \boldsymbol{Y}$$

with $\nabla \boldsymbol{Y}$ a $\binom{1}{1}$ tensor. The definition of ∇ on general tensors is obtained by requiring it to satisfy Leibniz on general tensor products

$$\nabla(\boldsymbol{S} \otimes \boldsymbol{T}) = \nabla \boldsymbol{S} \otimes \boldsymbol{T} + \boldsymbol{S} \otimes \nabla \boldsymbol{T}$$

The action on 1-forms follows from Leibniz

$$\nabla_{\boldsymbol{X}}(\boldsymbol{\Omega}(\boldsymbol{Y})) = (\nabla_{\boldsymbol{X}} \boldsymbol{\Omega})(\boldsymbol{Y}) + \boldsymbol{\Omega}(\nabla_{\boldsymbol{X}} \boldsymbol{Y})$$

in terms of a local basis $\{\boldsymbol{E}_a\}$ and $\{\boldsymbol{e}^a\}$ we have

$$\nabla_{\boldsymbol{X}}(\Omega_a Y^a) = (\nabla_{\boldsymbol{X}} \boldsymbol{\Omega})_a Y^a + \Omega_a (\nabla_{\boldsymbol{X}} \boldsymbol{Y})^a$$

since $\nabla_{\boldsymbol{X}} \boldsymbol{Y} = \nabla_{\boldsymbol{X}}(Y^a \boldsymbol{E}_a) = (\boldsymbol{X}(Y^a) + Y^b \boldsymbol{\omega}^a{}_b(\boldsymbol{X}))\boldsymbol{E}_a$, then

$$(\nabla_{\boldsymbol{X}} \boldsymbol{\Omega})_a = \boldsymbol{X}(\Omega_a) - \Omega_b\, \boldsymbol{\omega}^b{}_a(\boldsymbol{X})$$

For $\boldsymbol{\Omega} = \boldsymbol{e}^b$ we conclude that

$$\nabla_{\boldsymbol{E}_a} \boldsymbol{e}^b = -\boldsymbol{\omega}^b{}_m(\boldsymbol{E}_a)\, \boldsymbol{e}^m = -\omega_a{}^b{}_m\, \boldsymbol{e}^m$$

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Hadamard formula:

$$e^A B e^{-A} = e^{[A,} B \tag{0.1}$$

here the exponential is understood as $e^{[A,} \equiv (1 + [A, \cdot] + \frac{1}{2}[A, [A, \cdot]] + \ldots)$. Then,

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \ldots \tag{0.2}$$

when thinking of this expression in terms of matrix representation of groups, it means that conjugation by a group element is closed on the Lie algebra.

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`Proof`: consider $f(s) = e^{sA}Be^{-sA}$, then

$$\frac{df}{ds} = e^{sA}ABe^{-sA} + e^{sA}B(-A)e^{-sA} = Af(s) - f(s)A = [A, f]$$

From this we find $\ddot{f} = [A, \dot{f}] = [A, [A, f]]$, ... $f^{(n)} = [A, [A..[A, f]]..]$ with n commutators. If we evaluate these expressions at zero and use $f(0) = B$, we obtain $f^{(n)}(0) = [A, [A, ..[A, B]]..]$, then

$$f(s) = B + s[A, B] + \frac{s^2}{2}[A, [A, B]] + \frac{s^3}{3!}[A, [A, [A, B]]]....$$

—

Duhamel formula:

$$\delta e^Z = e^Z \int_0^1 ds\, e^{-sZ}\delta Z e^{sZ} \tag{0.3}$$

This formula is particularly useful for computing derivatives $\delta = \partial_t$ with $Z(t)$ a matrix. Where do we place the derivative $Z'(t)$ in $e^{Z(t)}$? Everywhere, this is, in all the positions in the expansion!

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`Proof`: same trick, take $f(s) = e^{-sZ}\vec{\Delta}(e^{sZ})$ with $\vec{\Delta}$ an operator acting on anything to its right. Then,

$$\frac{df}{ds} = e^{-sZ}(-Z)\vec{\Delta}(e^{sZ}) + e^{-sZ}\vec{\Delta}(Ze^{sZ}) = e^{-sZ}[\vec{\Delta}, Z]e^{sZ}$$

Integrating both sides gives

$$f(1) - f(0) = \int_0^1 ds\, e^{-sZ}[\vec{\Delta}, Z]e^{sZ}$$

The lhs can be worked out to give

$$f(1) - f(0) = e^{-Z}\vec{\Delta}\, e^Z - \vec{\Delta} = e^{-Z}(\vec{\Delta}\, e^Z - e^Z\vec{\Delta}) = e^{-Z}[\vec{\Delta}, e^Z]$$

inserting above we find

$$[\vec{\Delta}, e^Z] = e^Z \int_0^1 ds\, e^{-sZ}[\vec{\Delta}, Z]e^{sZ}$$

Calling $\delta e^Z = [\vec{\Delta}, e^Z]$ we get (0.3).

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Rewrite the conjugation on the rhs using (0.1) and the definition (0.10)

$$\delta e^Z = e^Z \int_0^1 ds\, e^{-s\,ad_Z}\,\delta Z$$

the s-integration on the rhs gives

$$\delta e^Z = e^Z \left. \frac{e^{-s\,ad_Z}}{-ad_Z}\right|_0^1 \delta Z$$

we conclude that

$$e^{-Z}\delta e^Z = \frac{1-e^{-\,ad_Z}}{ad_Z}\delta Z \tag{0.4}$$

The rhs should be understood as the expansion $\sum_{k=0} \frac{(-ad_Z)^k}{(k+1)!}$. The nested commutators show that the left invariant form on the lhs is Lie algebra valued.

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Left invariant forms belong to the Lie algebra: (0.4) can be alternatively obtained in the following way: consider $g(s,t) = e^{sZ(t)}$. By the definition of exponential,

$$g^{-1}\frac{\partial g}{\partial s} = Z(t)$$

Defining

$$\begin{aligned} B(s,t) = g^{-1}\frac{\partial g}{\partial t} &= e^{-sZ(t)}\frac{\partial e^{sZ(t)}}{\partial t} \\ \Rightarrow \quad \partial_s B &= -ZB + g^{-1}\partial_t(gZ) \\ \Rightarrow \quad \partial_s B &= -ZB + BZ + \dot{Z} \end{aligned} \tag{0.5}$$

we then find that B satisfies

$$\frac{\partial B}{\partial s} = -[Z,B] + \dot{Z} \quad \text{with b.c.} \quad B(0,t) = 0$$

Solving in power series in s

$$\begin{aligned} B(s,t) &= s\dot{Z} + \frac{s^2}{2!}(-ad_Z)\dot{Z} + ... + \frac{s^n}{n!}(-ad_Z)^{n-1}\dot{Z} + ... \\ &= s\,\phi(-s\,ad_Z)\dot{Z} \end{aligned}$$

where $\phi(z) = \frac{e^z - 1}{z}$. Setting $s = 1$ we reobtained (0.4).

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Baker-Campbell-Hausdorff formula: how to write the product of two exponentials as a single one?

$$e^A e^B = e^Z$$

then

$$Z = A + \left(\int_0^1 ds \, \psi(e^{[A,} \; e^{s[B,}) \right) B \tag{0.6}$$

with $\psi(x) = \frac{x \ln x}{x-1}$ the generating function of Bernouilli numbers: $\psi(e^{-y}) = \sum_{n=0}^{\infty} B_n \frac{y^n}{n!}$. The expression is formal and has a finite radius of convergence. In the case of non-compact Lie groups if A, B are far away from the origin the series on the rhs diverges. The construction of Z in terms of nested commutators shows, for the case of Lie groups, that Z belongs to the Lie algebra. An explicit expansion with all numerical coefficients was given by Eugene Dynkin in 1947 (see wikipedia).

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`Proof`: consider $e^{Z(s)} = e^A e^{sB}$ then for $\delta = \partial_s$

$$\delta e^{Z(s)} = e^A B e^{sB} = e^Z B \quad \Rightarrow \quad B = e^{-Z} \delta e^Z = \frac{1 - e^{-\, ad_Z}}{ad_Z} \delta Z$$

where we used (0.4). Then

$$\begin{aligned} Z'(s) &= \frac{ad_Z}{1 - e^{-\, ad_Z}} B \\ &= \psi(e^{[Z,}) B \end{aligned} \tag{0.7}$$

where we defined

$$\psi(x) \equiv \frac{x \ln x}{x - 1} = 1 - \sum_{n=1}^{\infty} \frac{(1-x)^n}{n(n+1)}$$

Now, from (0.1)

$$\begin{aligned} e^{[Z,} X &= e^Z X e^{-Z} \\ &= e^A e^{sB} X e^{-sB} e^{-A} \\ &= e^A (e^{s[B,} X) e^{-A} \\ &= e^{[A,} e^{s[B,} X \end{aligned}$$

Inserting this in (0.7) and performing an s-integration as above we find

$$Z(1) - Z(0) = \int_0^1 ds\, \psi(e^{[A,}e^{s[B,})B$$

from $Z(0) = A$ we get (0.6).

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The first few terms of the expansion are

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])-\frac{1}{24}[B,[A,[A,B]]]+\dots}$$

—

Group theory conventions and definitions:

. Group element: $g \in G$. Near the identity $g = \exp(X)$ with $X \in \mathfrak{g} = Lie(G)$

. Lie algebra generators: $[T_a, T_b] = f_a{}^c{}_b T_c$, any $X = X^a T_a$, $a = 1, \ldots n$. Antisymmetry of the commutator implies

$$f_a{}^c{}_b = -f_b{}^c{}_a \tag{0.8}$$

. adjoint representation of Lie algebra: A linear transformation acting on the Lie algebra vector space can be naturally associated to any X on the Lie algebra as:

$$ad_X Y \equiv [X, Y], \quad \forall\, Y \in \mathfrak{g}.$$

Jacobi identity implies that

$$[ad_X, ad_Y] = ad_{[X,Y]} \tag{0.9}$$

Then, the map $X \to ad_X$ is a representation of the algebra. The matrix representatives of the generators in T_a basis are:

$$(T_a^{(adj)})^m_n = f_a{}^m{}_n.$$

. Adjoint action: action of the group on the Lie algebra

$$Ad_g Y \equiv g Y g^{-1}, \quad g \in G,\ Y \in \mathfrak{g}$$

Writing $g = e^{tX}$ one finds using (0.2)

$$Ad_{\exp(tX)} Y = e^{tX} Y e^{-tX} = Y + t[X, Y] + \frac{1}{2} t^2 [X, [X, Y]] + \ldots$$

The ad_X exponentiation gives

$$e^{ad_{tX}} Y \equiv Y + t[X, Y] + \frac{1}{2} t^2 [X, [X, Y]] + \ldots \tag{0.10}$$

Then,

$$Ad_{\exp(X)} = \exp(ad_X)$$

. Killing-Cartan form: the linear ad representation induces an inner product in $\mathfrak{g}$:

$$\langle X, Y\rangle \equiv -\mathsf{tr}[ad_X ad_Y]. \tag{0.11}$$

The expansion $X = X^a T_a$ reduces the computation of $\langle\ ,\ \rangle$ to the knowledge of the Killing-Cartan metric[1]

$$\mathfrak{K}_{ab} \equiv -\mathsf{tr}[T_a^{(adj)} T_b^{(adj)}] = -f_a{}^m{}_n\, f_b{}^n{}_m.$$

The inner product satisfies

$$\begin{aligned} \langle ad_X Y, Z\rangle + \langle Y, ad_X Z\rangle &= \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle \\ &= -\mathsf{tr}[ad_{[X,Y]} ad_Z] - \mathsf{tr}[ad_Y ad_{[X,Z]}] = 0 \end{aligned} \tag{0.12}$$

in the last equality we used (0.9), cyclicity of trace and Jacobi identity. (0.12) is the infinitesimal expression of the inner product's invariance under the Adjoint action

$$\langle \exp(ad_{tX})Y, \exp(ad_{tX})Z\rangle = \langle Y, Z\rangle. \tag{0.13}$$

Returning to (0.11) and inserting the generators T_a one finds

$$\begin{aligned} 0 &= \langle ad_{T_a} T_b, T_c\rangle + \langle T_b, ad_{T_a} T_c\rangle \\ &= \langle [T_a, T_b], T_c\rangle + \langle T_b, [T_a, T_c]\rangle \\ &= f_a{}^m{}_b \langle T_m, T_c\rangle + f_a{}^m{}_c \langle T_b, T_m\rangle \\ &= f_a{}^m{}_b\, \mathfrak{K}_{mc} + f_a{}^m{}_c\, \mathfrak{K}_{bm} \end{aligned} \tag{0.14}$$

Lowering the indices with the Killing-Cartan metric $f_{anb} \equiv \mathfrak{K}_{nm}\, f_a{}^m{}_b$, then

$$f_{acb} + f_{abc} = 0 \quad \Rightarrow \quad f_{abc} = -f_{acb}$$

This relation and (0.8) imply totally antisymmetric structure constants

$$f_{abc} = -f_{cba} = -f_{acb} \tag{0.15}$$

The inverse (symmetric) metric satisfies $\mathfrak{K}^{ac}\mathfrak{K}_{cb} = \delta^a_b$. In particular (0.15) im-

[1]For the case of compact semisimple groups, by appropriately normalizing the generators we can set $K_{ab} = \delta_{ab}$.

plies

$$f_{m\ a}^{\ m} = \mathfrak{K}^{mn} f_{mna} = 0 \tag{0.16}$$

. Left invariant forms: from the group element $g(\xi)$ we construct:

$$g^{-1}\boldsymbol{d}g = \boldsymbol{e}^a T_a, \qquad \boldsymbol{e}^a = e^a_\mu(\xi) d\xi^\mu \tag{0.17}$$

(0.4) ensures it belongs to $\mathfrak{g}$. The set $\{e^a\}$ provide a globally defined 1-form frames for the cotangent space $T^\star\mathcal{M}$. The name left invariant comes from invariance of $\boldsymbol{e}^a$ under constant left shifts $g \to g_L\, g$. Notice that under right shifts the *currents* transform in the adjoint: $g \to g\, g_R$ imply $\boldsymbol{e}^a \to g_R^{-1}\boldsymbol{e}^a g_R$.

. bi-invariant Killing metric over G:

$$\mathsf{g} = ds^2 = -\mathsf{tr}[g^{-1}\boldsymbol{d}g \otimes g^{-1}\boldsymbol{d}g] \;\Rightarrow\; g_{\mu\nu}(\xi) = e^a_\mu(\xi) e^b_\nu(\xi) \mathfrak{K}_{ab} \tag{0.18}$$

For an appropriate choice of generators the Killing form can be put in the form $\mathfrak{K}_{ab} = \delta_{ab}$, then we can interpret the left invariant forms $\boldsymbol{e}^a$ in (0.18) as vierbeins over G. We then write

$$\mathsf{g} = \mathfrak{K}_{mn}\, \boldsymbol{e}^m \otimes \boldsymbol{e}^n \tag{0.19}$$

The name bi-invariant comes from the invariance of the metric under left and right shifts. Left invariance is immediate and right invariance follows from (0.13). These two invariances implies $G \times G$ isometry group for g.

. Left invariant vector fields: define

$$E^\mu_a \equiv \mathfrak{K}_{ab}\, g^{\mu\nu} e^b_\nu \quad \Rightarrow \quad E^\nu_a e^a_\mu = \delta^\nu_\mu \quad \text{and} \quad e^a_\mu E^\mu_b = \delta^a_b \tag{0.20}$$

They provide a non-holonomic basis for the tangent bundle $T\mathcal{M}$

$$\boldsymbol{E}_a \equiv E^\mu_a \boldsymbol{\partial}_\mu$$

Frames and vectors are dual to each other

$$\boldsymbol{e}^a(\boldsymbol{E}_b) = \langle \boldsymbol{e}^a | \boldsymbol{E}_b \rangle = \delta^a_b$$

We denote the action of g on vectors $\boldsymbol{U} = U^a \boldsymbol{E}_a$ as

$$\mathsf{g}(\boldsymbol{U},\boldsymbol{V}) = \langle \mathsf{g}|\boldsymbol{U},\boldsymbol{V}\rangle = \mathfrak{K}_{mn}\,(\boldsymbol{e}^m\otimes\boldsymbol{e}^n)(\boldsymbol{U},\boldsymbol{V}) = \mathfrak{K}_{mn}\boldsymbol{e}^m(\boldsymbol{U})\,\boldsymbol{e}^n(\boldsymbol{V}) = \mathfrak{K}_{mn}U^m U^n$$

. Maurer-Cartan identity: The left invariant forms satisfy

$$\boldsymbol{d}(g^{-1}\boldsymbol{d}g) + g^{-1}\boldsymbol{d}g \wedge g^{-1}\boldsymbol{d}g = 0 \tag{0.21}$$

Calling $\boldsymbol{A} = g^{-1}\boldsymbol{d}g$, this expression can be read as saying that the $\mathfrak{g}$-connection A is flat over the group manifold $\boldsymbol{F}(\boldsymbol{A}) = \boldsymbol{dA} + \boldsymbol{A}^2 = 0$. Writing (0.21) using (0.17) we find

$$\begin{aligned} T_a\,\boldsymbol{de}^a + T_b T_c\,\boldsymbol{e}^b \wedge \boldsymbol{e}^c &= 0 \\ T_a \boldsymbol{de}^a + \frac{1}{2}[T_b, T_c]\boldsymbol{e}^b \wedge \boldsymbol{e}^c &= 0 \\ T_a\left(\boldsymbol{de}^a + \frac{1}{2} f_b{}^a{}_c\,\boldsymbol{e}^b \wedge \boldsymbol{e}^c\right) &= 0 \end{aligned} \tag{0.22}$$

In components we have

$$\partial_\mu e^a_\nu - \partial_\nu e^a_\mu + f_b{}^a{}_c\,e^b_\mu e^c_\nu = 0 \tag{0.23}$$

With this expressions we can check that

$$[\boldsymbol{E}_a, \boldsymbol{E}_b] = f_a{}^c{}_b \boldsymbol{E}_c$$

with constant structure constants. Then, calling $\boldsymbol{E}_a \to \boldsymbol{L}_a$ we find left invariant vector fields over the group manifold G. They provide a realization of the Lie algebra as first order differential operators acting on G. The fact of being globally defined makes group manifolds parallelizable.

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`Proof`: from (0.20) we get

$$\partial_\nu(e^a_\mu E^\mu_b) = 0 \quad \Rightarrow \quad \partial_\nu E^\rho_b = -E^\rho_a\,\partial_\nu e^a_\mu\,E^\mu_a$$

The commutator takes the form

$$[E^\mu_a \boldsymbol{\partial}_\mu, E^\nu_b \boldsymbol{\partial}_\nu] = (E^\nu_a \partial_\nu E^\mu_b - E^\nu_b \partial_\nu E^\mu_a)\boldsymbol{\partial}_\mu$$

$$
\begin{aligned}
&= (E_a^\rho E_b^\nu - E_a^\nu E_b^\rho)\partial_\nu e_\rho^c \boldsymbol{E}_c \\
&= E_a^\rho E_b^\nu (\partial_\nu e_\rho^c - \partial_\rho e_\nu^c)\boldsymbol{E}_c \\
&= -E_a^\rho E_b^\nu f_m{}^c{}_n e_\nu^m e_\rho^n \boldsymbol{E}_c \\
&= f_a{}^c{}_b \, \boldsymbol{E}_c
\end{aligned}
\tag{0.24}
$$

in going from the third to the fourth line we used (0.23), from the fourth to the last we used (0.20) and the antisymmetry of structure constants.

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In the basis $\{\boldsymbol{L}_a\}$ the metric components are constant $g_{ab} = \mathsf{g}(\boldsymbol{L}_a, \boldsymbol{L}_b) = \mathfrak{K}_{ab}$. The isometry group of the Killing metric $G \times G$ arises from the left and right invariant vector fields which close to give $\mathfrak{g}$ as

$$
[\boldsymbol{L}_a, \boldsymbol{L}_b] = f_a{}^c{}_b \boldsymbol{L}_c, \qquad [\boldsymbol{R}_a, \boldsymbol{R}_b] = -f_a{}^c{}_b \boldsymbol{R}_c, \qquad [\boldsymbol{L}_a, \boldsymbol{R}_b] = 0
$$

The sign change in the right invariant vector fields arises from the sign shift in the Maurer-Cartan equation for the right invariant forms $\boldsymbol{d}g\, g^{-1} = \boldsymbol{f}^a\, T_a$, i.e.

$$
\boldsymbol{d}(\boldsymbol{d}g\, g^{-1}) - \boldsymbol{d}g\, g^{-1} \wedge \boldsymbol{d}g\, g^{-1} = 0
$$

. Spin and Affine connection on G: (0.22) implies that the choice

$$
\boldsymbol{\omega}^a{}_b = \frac{1}{2} f_m{}^a{}_b \, \boldsymbol{e}^m \tag{0.25}
$$

is a natural torsion free spin connection over the group manifold. The parallel transport over the manifold is determined by the spin connection

$$
\nabla_{\boldsymbol{E}_a} \boldsymbol{E}_b = \omega_a{}^c{}_b \, \boldsymbol{E}_c, \qquad \nabla_{\boldsymbol{E}_a} \boldsymbol{e}^m = -\omega_a{}^m{}_b \, \boldsymbol{e}^b
$$

In the particular case of a group manifold we choose the basis E_a to coincide with the left invariant vector fields $\boldsymbol{E}_a = \boldsymbol{L}_a$[2]. We can then reinterpret the choice (0.25) as follows: the structure provided by G allows to define a 1-parameter family of connections

$$
\nabla^{(\lambda)}_{\boldsymbol{L}_a} \boldsymbol{L}_b = \lambda [\boldsymbol{L}_a, \boldsymbol{L}_b] = \lambda f_a{}^c{}_b \, \boldsymbol{L}_c \tag{0.26}
$$

[2]We always have two possible global basis, i.e. $\{\boldsymbol{L}_a\}$ and $\{\boldsymbol{R}_a\}$.

This λ-family of connections is compatible with the Killing metric (0.18) since

$$\begin{aligned}
\nabla_a^{(\lambda)}\mathrm{g} &= \nabla_a^{(\lambda)}\mathfrak{K}_{mn}\,\boldsymbol{e}^m\otimes\boldsymbol{e}^n + \mathfrak{K}_{mn}\nabla_a^{(\lambda)}\boldsymbol{e}^m\otimes\boldsymbol{e}^n + \mathfrak{K}_{mn}\boldsymbol{e}^m\otimes\nabla_a^{(\lambda)}\boldsymbol{e}^n \\
&= -\mathfrak{K}_{mn}\left(\omega_a{}^m{}_b\,\boldsymbol{e}^b\otimes\boldsymbol{e}^n + \boldsymbol{e}^m\otimes\omega_a{}^n{}_b\,\boldsymbol{e}^b\right) \\
&= -\lambda f_{anb}\left(\boldsymbol{e}^b\otimes\boldsymbol{e}^n + \boldsymbol{e}^n\otimes\boldsymbol{e}^b\right) = 0
\end{aligned} \tag{0.27}$$

the first term in the first line is zero since $\mathfrak{K}_{ab}$ are constants and the vanishing in the last line follows from the antisymmetry of the structure constants, see (0.15).

Among the whole λ-family (0.26), the choice $\lambda = 1/2$ is singled out for being torsion free (cf. (0.22)). Since there is a unique torsion free metric compatible connection, then $\nabla_a^{(1/2)}$ is the Levi-Civita connection of the metric (0.18).

. Riemann curvature tensor: for the Levi-Civita connection (0.25)

$$\begin{aligned}
\boldsymbol{R}^a{}_b &\equiv \boldsymbol{d}\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c\wedge\boldsymbol{\omega}^c{}_b \\
&= \frac{1}{2}f_c{}^a{}_b\,\boldsymbol{d}\boldsymbol{e}^c + \frac{1}{4}f_m{}^a{}_c\,f_n{}^c{}_b\boldsymbol{e}^m\wedge\boldsymbol{e}^n \\
&= \left(-\frac{1}{4}f_c{}^a{}_b\,f_m{}^c{}_n + \frac{1}{4}f_m{}^a{}_c\,f_n{}^c{}_b\right)\boldsymbol{e}^m\wedge\boldsymbol{e}^n
\end{aligned} \tag{0.28}$$

. Ricci tensor: we can compute it in different component frames:

$$R_{b\nu} \equiv E_a^\mu R^a{}_{b\mu\nu}, \qquad R_{ab} = E_b^\nu R_{a\nu}$$

from the expression (0.28) one finds

$$R_{ab} = \frac{\mathfrak{K}_{ab}}{4} \quad \text{or} \quad R_{\mu\nu} = \frac{g_{\mu\nu}}{4}$$

Manifesting the fact of the geometry being Einstein.

. Scalar Curvature: being the group a homogeneous space the scalr curvature is constant

$$R \equiv \mathfrak{K}^{ab}R_{ab} = \frac{n}{4}$$

with n the group dimension[3].

[3]One might wonder whether the final result $R = n/4$ depends on the normalization of the generators, the answer is no: any change will scale $\mathfrak{K}$ which compensates upon contracting with its inverse. The $1/4$ factor is inherited from the $1/2$ in the spin connection.

. Laplace-Beltrami and quadratic Casimir: we are in position to construct three 2nd order operators. The Laplacian

$$\begin{aligned}\Delta &= g^{\mu\nu}\nabla_\mu\nabla_\nu \\ &= \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu) \\ &= g^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu})\partial_\nu\end{aligned} \tag{0.29}$$

and the quadratic Casimirs for the Left and Right actions. For concreteness consider

$$\begin{aligned}\mathcal{C}_L &= \mathfrak{K}^{ab}L_aL_b \\ &= \mathfrak{K}^{ab}E^\mu_a\partial_\mu(E^\nu_b\partial_\nu) \\ &= \mathfrak{K}^{ab}E^\mu_aE^\nu_b\partial_\mu\partial_\nu + \mathfrak{K}^{ab}E^\mu_a\partial_\mu(E^\nu_b)\partial_\nu\end{aligned} \tag{0.30}$$

The first terms in the last lines of (0.29) and (0.30) coincide since $g^{\mu\nu} = \mathfrak{K}^{ab}E^\mu_aE^\nu_b$. A manipulation of the second terms shows they coincide[4]. We conclude that on scalar functions

$$\Delta = \mathcal{C}_L = \mathcal{C}_R$$

—

`Proof`: we start from the last term in (0.29). Calling $e = \det e^a_\mu = \exp(\mathtt{tr}\ln e^a_\mu)$, we have $\partial_\mu e = e\,E^\nu_a\,\partial_\mu e^a_\nu$, then

$$\begin{aligned}\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}) &= \frac{\mathfrak{K}^{ab}}{e}\partial_\mu(eE^\mu_aE^\nu_b) \\ &= \mathfrak{K}^{ab}(E^\rho_c\,\partial_\mu e^c_\rho\,E^\mu_aE^\nu_b - E^\mu_c\,\partial_\mu e^c_\rho\,E^\rho_aE^\nu_b + E^\mu_a\partial_\mu E^\nu_b) \\ &= \mathfrak{K}^{ab}(E^\rho_c(\partial_\mu e^c_\rho - \partial_\rho e^c_\mu + \partial_\rho e^c_\mu)E^\mu_aE^\nu_b - E^\mu_c\,\partial_\mu e^c_\rho\,E^\rho_aE^\nu_b + E^\mu_a\partial_\mu E^\nu_b) \\ &= \mathfrak{K}^{ab}(E^\rho_c(-f_m{}^c{}_n e^m_\mu e^n_\rho) + E^\rho_c\partial_\rho e^c_\mu E^\mu_aE^\nu_b - E^\mu_c\,\partial_\mu e^c_\rho\,E^\rho_aE^\nu_b + E^\mu_a\partial_\mu E^\nu_b) \\ &= \mathfrak{K}^{ab}E^\mu_a\partial_\mu E^\nu_b\end{aligned} \tag{0.31}$$

[4] Working with the right invariant vector fields we arrive to the same result

$$\mathcal{C}_R = \mathfrak{K}^{ab}R_aR_b = \Delta$$

In going from the second to the third line we used the torsion free condition (0.23). The first term in the third line vanishes since it reduces to $f_{m\ c}^{\ c} = 0$ by (0.16) and second and third terms in the same line cancel mutually leading to the answer in the forth line.

—

. Left invariant vector field and right actions: L_a as a differential operator implements the right action

$$e^{\eta^a L_a} g(\xi) = g(\xi) g(\eta) = g(\zeta)$$

At the infinitesimal level $\eta \to 0$, acting on a representation $D^R(g) = e^{X^a T_a^{(R)}}$ we get

$$L_a D(g) = D(g) T_a^{(R)} \tag{0.32}$$

. Eigenfunctions of the Laplacian:

▷ *The matrix elements of the irreducible representations D^J are eigenfunctions of the Laplacian*

$$\Delta D^J(g(\xi)) = \lambda_J D^J(g(\xi))$$

▷ *The eigenvalue equals the Casimir of the irrep*

$$\lambda_J = \mathcal{C}(J) = \mathfrak{K}^{ab} T_a^{(J)} T_b^{(J)}$$

▷ *The $G \times G$ symmetry group of the group manifold is realized on the Laplacian eigenfunctions eigenspace with*

$$\Delta = \mathcal{C}_L = \mathcal{C}_R$$

▷ *The eigenspace degeneracy is $(d_J)^2$ with d_J the dimension of the D^J matrix.*

—

`Proof`: writing $D^J(g(\xi)) = e^{X^a(\xi) T_a^{(J)}}$ for the irrep J

$$\begin{aligned} \Delta D^J(g(\xi)) &= \mathfrak{K}^{ab} L_a L_b D^J(g(\xi)) = \mathfrak{K}^{ab} L_a D^J(g(\xi)) T_b^{(J)} \\ &= \mathfrak{K}^{ab} D^J(g(\xi)) T_a^{(J)} T_b^{(J)} = \lambda_J D^J(g(\xi)) \end{aligned} \tag{0.33}$$

where $\lambda_J = \mathfrak{K}^{ab} T_a^{(J)} T_b^{(J)}$ is the quadratic Casimir of the irrep J.

—

Eg: For $G = SU(2)$ with hermitic generators $L_a \to iL_a$ and normalizing $[L_a, L_b] = i\epsilon_{abc} L_c$ we have

$$\vec{L}^2 |j,m,k\rangle = \vec{R}^2 |j,m,k\rangle = j(j+1)|j,m,k\rangle$$

$$L_3 |j,m,k\rangle = m|j,m,k\rangle$$

$$R_3 |j,m,k\rangle = k|j,m,k\rangle$$

The fact of the Casimir having the same value for left and right symmetries arises from bi-invariance. The energy eigenstates are

$$H|j,m,k\rangle = \frac{j(j+1)}{4}|j,m,k\rangle$$

We can figure out the energy level degeneracy immediately because the energy levels only depend on j. There are $2j+1$ possible m values and $2j+1$ possible k values for each value of j, thus the total degeneracy is $(2j+1)^2$.

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