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Basic Units and Introduction to Natural Units

1 Basic units in particle physics

In particle physics, the preferred length unit is the femtometer (or fermi), where 1 fm = 10^{-15} m. For example, the proton radius is ~1.0 fm. Cross sections are typically measured in "barns", where 1b = 10^{-28} m^2. Energies are measured in GeV, or giga-electron volts (1 $\text{GeV} = 1.6\times10^{-10}$ J).

In particle physics, even the barn is huge (it was defined for low energy nuclear physics) and we more commonly use units such as mb (10^{-3} b), μb (10^{-6} b), nb (10^{-9} b), pb (10^{-12} b) and fb (10^{-15} b). The radius of the proton thus corresponds to a cross section of $\sigma = \pi r_p^2 = 31\text{mb}$. The cross section of W production in pp collisions at LHC energies is typically ~1 nb.

2 Natural units

In this course, we follow researchers in particle physics, nuclear physics and astrophysics in adopting "natural units", where $\hbar = 1$ and c = 1 and the unit of energy is the GeV. All basic quantities (length, area, time, rate, momentum, mass) can be expressed in terms of powers of GeV. Since $\hbar$ has units GeV-sec and c has units m/s, it is always possible to convert an expression for one of these quantities derived using natural units to "correct" form by putting in appropriate factors of $\hbar$ and c. We explore these ideas in the next few sections.

3 Natural units in energy, mass and momentum

For any system the total energy E, momentum p and mass m are related by the relativistic formula $E^2 = (pc)^2 + (mc^2)^2$. Thus E, pc and mc^2 have dimensions of energy, or GeV. Choosing units where c = 1, the energy relationship can be written in the simpler form $E^2 = p^2 + m^2$, with all quantities measured in GeV. Note that in published papers momenta and masses are always expressed as GeV/c and GeV/c^2, respectively. These relations are strictly true regardless of units.

For example, consider a system with E = 5 GeV, p = 4 GeV and m = 3 GeV, which clearly satisfies $E^2 = p^2 + m^2$. We can easily convert all quantities to SI units by using the energy conversion and c factors:

- $E = 5\,\text{GeV} = 5\cdot\left(1.6\times10^{-10}\right) = 8.0\times10^{-10}\,\text{J}$
- $p = 4\,\text{GeV}/c = 4\cdot\left(1.6\times10^{-10}\right)/\left(3\times10^{8}\right) = 2.13\times10^{-18}\ \text{kg}\cdot\text{m/s}$
- $m = 3\,\text{GeV}/c^2 = 3\cdot\left(1.6\times10^{-10}\right)/\left(3\times10^{8}\right)^2 = 5.33\times10^{-27}\ \text{kg}$

As a second example, let's find the momentum and kinetic energy in GeV units of a proton (mass $m_p = 1.67 \times 10^{-27}$ kg or $m_p c^2 = 0.938$ GeV) moving with a velocity of $0.05c$ (nonrelativistically).

Using natural units, we do the calculation expressing masses in GeV and velocities in units of c.

- $v = 0.05$
- $m_p = 0.938$ GeV
- $p = m_p v = 0.938 \times 0.05 = 0.0469$ GeV $= 46.9$ MeV
- $K = \frac{1}{2} m_p v^2 = \frac{1}{2}(0.938)(0.05)^2 = 0.00117$ GeV = 1.17 MeV

Now use the standard formulas using mass, velocity and c explicitly. This shows why the natural units make sense.

- $v = 0.05c$ or $\beta = v / c = 0.05$
- $m_p c^2 = 0.938$ GeV or $m_p = 0.938 \text{ GeV} / c^2$
- $p = m_p v = m_p c^2 (v / c) / c = 0.938 \times 0.05 / c = 0.0469 \text{ GeV} / \text{c}$
- $K = \frac{1}{2} m_p v^2 = \frac{1}{2}(m_p c^2)(v / c)^2 = \frac{1}{2}(0.938)(0.05)^2 = 0.00117$ GeV

These formulas utilize $m_p c^2$ explicitly, making the calculations in energy units more understandable.

4 Length and other units

For electromagnetic units, we start with the dimensionless fine structure constant α, where $\alpha = e^2 / 4\pi\varepsilon_0 \hbar c \simeq 1/137$. We can choose $\varepsilon_0 = 1$ (and $\mu_0 = 1$ to keep $c = 1/\sqrt{\mu_0 \varepsilon_0} = 1$), which makes $e = \sqrt{4\pi\alpha} \simeq 0.303$ dimensionless in natural units.

Note that $\hbar$ has units of energy $\times$ time while c is length/time. We see that

- Mass, momentum, energy are measured in GeV ($m = m_{\text{NU}} / c^2$, $p = p_{\text{NU}} / c$)
- Length is measured in GeV^{-1} ($L_{\text{SI}} = L_{\text{NU}} \times \hbar c$)
- Cross section (area) is measured in GeV^{-2} ($\sigma_{\text{SI}} = \sigma_{\text{NU}} \times (\hbar c)^2$)
- Time is measured in GeV^{-1} ($t_{\text{SI}} = t_{\text{NU}} \times \hbar$)
- Reaction or decay rates (sec^{-1}) are measured in GeV ($r_{\text{SI}} = r_{\text{NU}} / \hbar$)

- Velocity is in units of c and dimensionless ($|v| \le 1$)
- Charge is dimensionless

5 Example 1: Simple conversions

Here are useful conversion factors that are needed to covert between SI and natural units:

- $1\text{ GeV} = 10^9\text{ eV} = 1.60\times10^{-10}\text{ J}$
- $c = 3.00\times10^8\text{ m/s}$
- $\hbar = 6.58\times10^{-25}\text{ GeV}\cdot\text{sec}$
- $\hbar c = 0.197\text{ GeV}\cdot\text{fm}$
- $(\hbar c)^2 = 0.0389\text{ GeV}^2\cdot\text{fm}^2 = 0.389\text{ GeV}^2\cdot\text{mb} = 389\text{ GeV}^2\cdot\mu\text{b}$

Particle physics calculations of cross section, rate and size are almost always determined in natural units, but the conversion to ordinary units merely involves multiplying or dividing the formula or value by some combination of $\hbar$ and c. Some examples:

- Cross section of a process: $\sigma = 10^{-3}\text{GeV}^{-2} = 10^{-3}\times(\hbar c)^2 = 0.389\mu\text{b}$
- Decay rate of ϕ meson: $\Gamma = 4.0\,\text{MeV} = 4.0\times10^{-3}/\hbar = 6.08\times10^{21}/\text{sec}$
- Length scale measured at LEP: $r = 0.01\text{GeV}^{-1} = 0.01\times\hbar c = 0.00197\,\text{fm}$
- $e^+e^-\to q\bar{q}$ cross section vs COM energy: $\sigma = 4\pi\alpha^2/27E^2 = 4\pi\alpha^2(\hbar c)^2/27E^2$

6 Example 2: Relativistic momentum and energy

Let's calculate various kinematic quantities for a Λ ($m_\Lambda = 1.116\text{GeV}/c^2$) moving with momentum $p_\Lambda = 2.1\text{GeV}/c$. We use the relativistic expressions $E = \sqrt{(pc)^2 + (mc^2)^2} = \gamma mc^2$ and $p = \gamma mv$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Using natural units, we get:

- Total energy: $E_\Lambda = \sqrt{p_\Lambda^2 + m_\Lambda^2} = \sqrt{2.1^2 + 1.116^2} = 2.378\text{GeV}$
- Kinetic energy: $K_\Lambda = E_\Lambda - m_\Lambda = 1.262\text{GeV}$
- Velocity: $v_\Lambda = p_\Lambda / E_\Lambda = 2.1/2.378 = 0.883$
- Gamma factor: $\gamma_\Lambda = E_\Lambda / m_\Lambda = 2.378/1.116 = 2.131$

With natural units, we can trivially determine that this particle is moving relativistically from the fact that its momentum is comparable to its mass.

7 Some more exercises

- Find the momentum in SI units for the Λ particle in the previous example.
- What is 10 MeV in natural units?
- What is 130,000 m/s in natural units?
- What is 1 fm in natural units?
- What is 1 fs in natural units?
- What is $\beta \equiv v / c$ and γ for a proton moving around the LHC with momentum 7 TeV?
- Write Coulomb's law for two elementary charges using α instead of e.
- The QCD potential between two quarks is approximately $V = \alpha_s / r + kr$ in natural units, where r is the qq separation, α_s is the dimensionless QCD coupling strength (analogous to α) and k is a constant with units GeV/fm. Write this formula in dimensionally correct form by inserting $\hbar$ and c factors in the appropriate places.
- The Planck length is defined in natural units as $\sqrt{G_N}$, where G_N is the Newtonian gravitational constant. Put in the correct factors of $\hbar$ and c to make it dimensionally correct. (Hint: find the units of G_N from the potential energy formula.) What is the Planck length in SI units? What is the corresponding Planck energy in GeV?