Drop formation in a falling stream of liquid

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The narrowing of a falling stream of liquid is a well-known demonstration of the equation of continuity. We consider the behavior of the bottom of a falling liquid stream where the stream ceases to narrow, and swells and forms droplets. Drop formation is demonstrated by detailed photos of experiments. A simple mathematical description of the observations is given, including the key processes responsible for drop formation.
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I. INTRODUCTION

Drop formation in a falling stream of liquid is an interesting and commonly observed phenomenon that has been a subject of much research from the beginnings of hydrodynamic theory. It soon was realized that surface tension is the driving force in drop formation, because it tends to reduce the surface area by decreasing the radius of the stream. In the 19th century the phenomenon was described by Rayleigh.[1–3] By using a stability analysis of an infinite cylinder of liquid with radius r_0, Rayleigh studied perturbations of different wavelengths and calculated their growth rates. Although long wavelength perturbations tend to form large drops with the smallest possible surface area, large mass transport between the drops is required. Surface tension and inertia strike a balance at the wavelength $\lambda = 9r_0$. In other words, drop formation depends crucially on the ratio between inertia and surface tension, which is known as the Weber number.[4]

Subsequently, the theory of drop formation was refined by including the viscosity,[5] higher order nonlinear effects,[6] and surface charges.[7] Recently, drop breakup studies have been devoted to the idea of scale invariance.[8–11] The break up process becomes scale invariant if the dynamics is governed by proximity to the thinnest point in the neck, where the drop eventually breaks apart. Here the fluid decreases in diameter until its thickness goes to zero, and, at some point, it has become so much smaller than any other macroscopic length in the system that the larger lengths no longer matter for a description of the neck itself. The dynamics depends only on the thickness of the neck and does not depend on the size of the nozzle or the size of the drop. Therefore, all memory of the initial and boundary conditions is lost. Some important exceptions to this type of behavior have recently been analyzed.[11]

The importance of visualization of drop formation has been understood from the outset of studies in this field. In 1833 Savart investigated the decay of fluid jets by illuminating a jet with sheets of light. He observed undulations growing on a jet of water that cause the breaking of the jet and subsequent drop formation.[8] The development of photography and the recent use of computers gave us new ability to visualize the dynamics of drop formation. The development of visualization from early photographs taken by Rayleigh[12] to the first high-resolution photographs of water falling from a faucet taken by Peregrine *et al*.[13] are reviewed in Ref. 8.

Although drop formation can be experimentally observed and analyzed, its quantitative mathematical description is difficult. The theory of drop formation based on the Navier-Stokes equations[3,8,14,15] is not accessible to undergraduates. Simplifications of the theory using one-dimensional approximations of the Navier-Stokes equations[14] are also too difficult.

In this paper we suggest a semi-quantitative approach to drop formation. The quantitative description involves only the key processes involved in drop formation in a falling stream of liquid, which require understanding about the concepts of surface tension, pressure in liquids, and conservation of volume flux in noncompressible liquids (the equation of continuity). Our study concerns Newtonian fluids for which previous experimental observations[10,16] have shown that the free-surface shapes of the fluid are very similar when approaching the pinch point. The dynamics become universal, that is, independent of initial conditions such as the nozzle radius and independent of the experimental method of forming the droplets,[8] whether it is by a jetting, liquid bridge, or dripping experiment (as performed in this study).

As a starting point, we consider the narrowing of a stream of water coming from a faucet. The narrowing of the stream of a nonviscous liquid represents an analytically solvable problem.[17,18] We are interested in the bottom part of the falling stream of water, where the transition from a thin stream into drops takes place.

II. NARROWING OF A FALLING STREAM OF LIQUID

The cross-section of a stream of liquid coming from a faucet becomes smaller due to the Earth's gravitation. For a falling stream of liquid with cross-section S_0 and velocity v_0 at the faucet (see Fig. 1), we can calculate the cross-section S and the velocity of water v at a time t later. According to the equation of continuity, the conservation of the volume flux Φ_V, we can write

$$\Phi_V = S_0 v_0 = S v. \tag{1}$$

Due to gravity, the velocity v changes with distance from the faucet, h:

$$v^2 = v_0^2 + 2gh. \tag{2}$$

The substitution of Eq. (2) into Eq. (1) gives

$$S = S_0 \frac{v_0}{\sqrt{v_0^2 + 2gh}}. \tag{3}$$

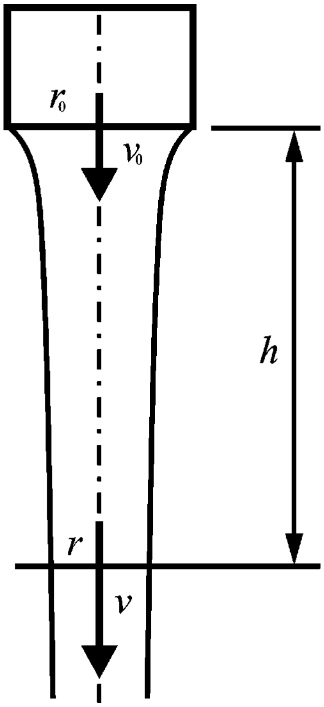

Fig. 1. Narrowing of a stream of liquid falling from a tap.

The radius of the stream of water, r, follows directly from Eq. (3) by taking into account the relations $S=\pi r^2$ and $S_0=\pi r_0^2$ (see Fig. 1):

$$r=r_0\sqrt[4]{\frac{v_0^2}{v_0^2+2gh}}. \tag{4}$$

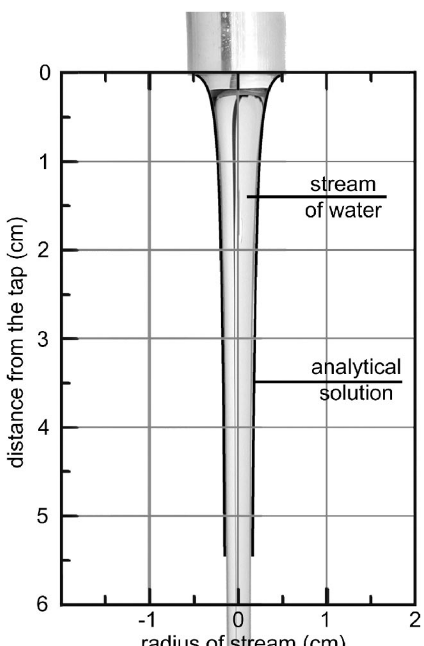

Fig. 2. Photo of a falling stream of water with the analytical solution of the form of the stream. Experimental measurements: $r_0=0.50\pm0.02$ cm, $v_0=11.0\pm0.5$ cm/s; the calculations were done with $r_0=0.50$ cm and $v_0=11.0$ cm/s.

Fig. 3. Drop formation at the bottom of a stream of water. The drops grow as swellings in the stream.

It is impressive how well Eq. (4) predicts the form of a real stream of water. Figure 2 shows a photo of a falling stream of water and the superimposed solution obtained from Eq. (4). The latter closely matches the form of the real stream.

To obtain the results in Fig. 2, the initial velocity of the stream, v_0, has to be measured very accurately. We propose an effective procedure in which v_0 can be accurately determined by measuring the volume flux of water Φ_V. We simply measure the volume V of water flowing into a glass and the time t taken to fill it. Because $\Phi_V=\text{const}$, it follows that $\Phi_V=V/t$. From the relation $\Phi_V=S_0v_0$, we can express v_0 as:

$$v_0=\frac{V}{S_0t}. \tag{5}$$

For the experimental system shown in Fig. 2, the initial velocity is $v_0=11.0\pm0.5$ cm/s. If we use this value in Eq. (4), the radius as a function of h of the stream of water can be calculated, and the results compared with experiment as shown in Fig. 2.

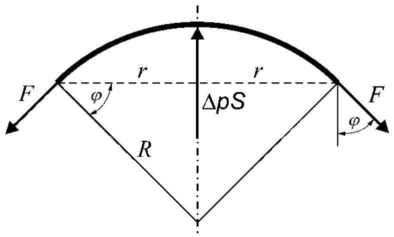

Fig. 4. Cross-section of the cylindrical stream of liquid. The pressure difference Δp acting on the surface S of the cylinder is in balance with the force of surface tension F.

III. DROP FORMATION

The narrowing of a falling stream of liquid discussed in Sec. II closely models the form of a real stream. However, this model holds only for the upper part of the stream. The bottom part of the stream where drops are formed requires additional explanation. In Fig. 3 we see that at the bottom of the stream, the stream does not become narrower; rather, it swells again.

At the top of the stream, perturbations in the stream cause waves of very small amplitude that cannot be easily observed. The undulations grow in time and cause drop formation at the bottom of the stream. This process is favored by surface tension because the surface area of the liquid tends to be reduced. It would be desirable to collect all the fluid into

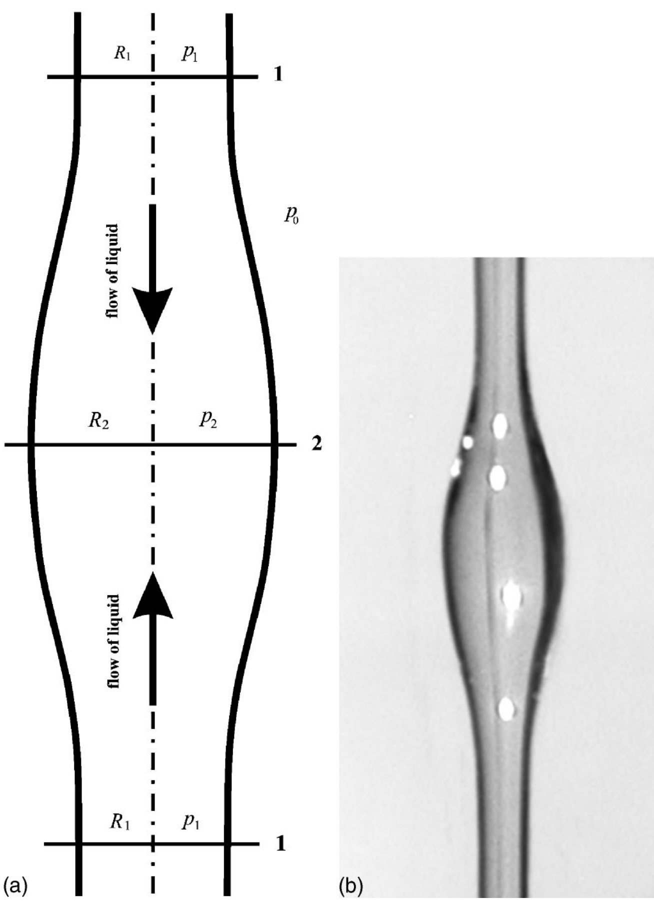

Fig. 5. Early phase of drop formation: (a) schematic presentation and (b) photograph. The liquid flows from a location with higher pressure p_1 into a swelling with lower pressure p_2.

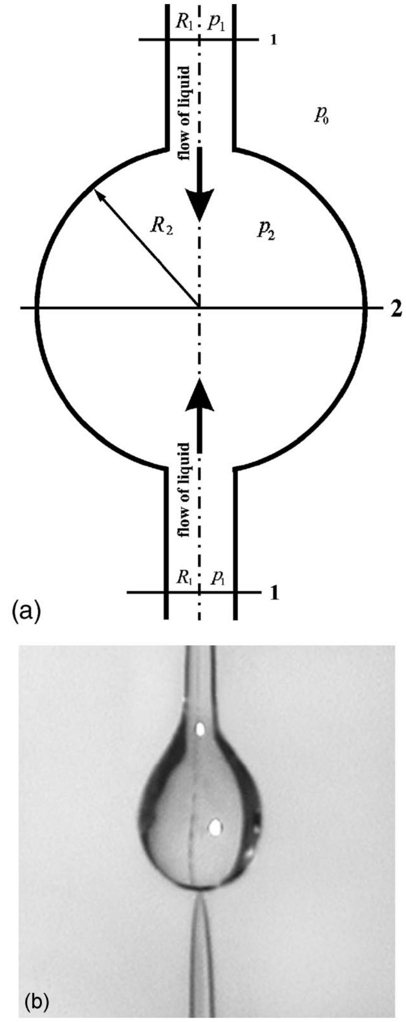

Fig. 6. End phase of drop formation: (a) schematic presentation and (b) photo. The liquid flows from an area of higher pressure p_1 into a spherical drop with lower pressure p_2.

one sphere, corresponding to the smallest surface area. However, the surface tension has to work against inertia, which opposes fluid motion over long distances. The two effects, surface tension and inertia, strike a balance and smaller drops are formed.

To understand the growth of undulations in a stream of water, we recall the expression for the pressure in a stream of liquid. We locally approximate the stream of water by a cylinder. Due to surface tension, the pressure in the cylinder is larger than in the surrounding air. The pressure difference between the pressure in the stream, p, and the surrounding air pressure, p_0, $\Delta p = p - p_0$, acting on the surface of the cylinder, S, is equal to the surface tension force F (see Fig. 4):

$$\Delta p S - 2F \cos \varphi = 0. \tag{6}$$

We write $S = 2rl$, where l is the length of the cylinder, and $F = \gamma l$, where γ is the surface tension, and express Δp as

$$\Delta p = \gamma \frac{\cos \varphi}{r}. \tag{7}$$

Because $r = R \cos \varphi$, Δp is given by

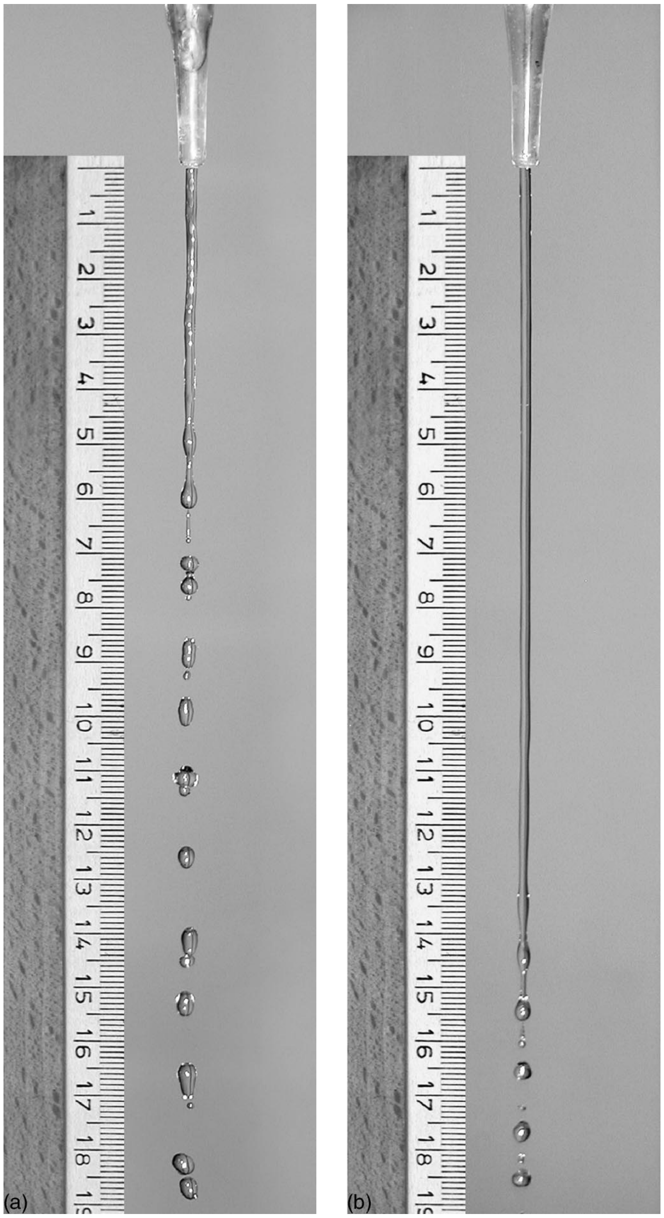

Fig. 7. Drop formation showing the dependence on the surface tension of liquid: (a) fresh water with surface tension 0.073 N/m (drops form at $h = 5 \pm 1$ cm) and (b) soapy water with surface tension 0.03 N/m (drops form at $h = 15 \pm 2$ cm). The experiment was done for $r_0 = 2.0 \pm 0.2$ mm and $v_0 = 5.0 \pm 0.5$ cm/s.

$$\Delta p = \gamma \frac{1}{R}. \tag{8}$$

Equation (8) shows that the pressure in the stream is larger where the stream has a smaller radius and vice versa. According to Eq. (8), the liquid flows from an environment of higher pressure p_1 into the swelling of the stream with lower pressure p_2 (see Fig. 5). The driving force, $p_1 - p_2$, for the flow of liquid into the swelling can be approximated by using Eq. (8) for both regions 1 and 2 (see Fig. 5):

$$p_1 - p_2 = \gamma \left(\frac{1}{R_1} - \frac{1}{R_2} \right). \tag{9}$$

Figure 5 shows that region 2 is not an ideal cylinder, and hence Eq. (9) is just a rough approximation for the driving force for the flow of liquid into the growing drop. For a more accurate estimation of the pressure in the growing drop, both radii of curvature have to be taken into account. In analogy to the derivation of Eq. (8), the pressure in the growing drop can be expressed more accurately by

$$p_2 - p_0 = \gamma \left(\frac{1}{R_{2,\mathrm{I}}} + \frac{1}{R_{2,\mathrm{II}}} \right), \tag{10}$$

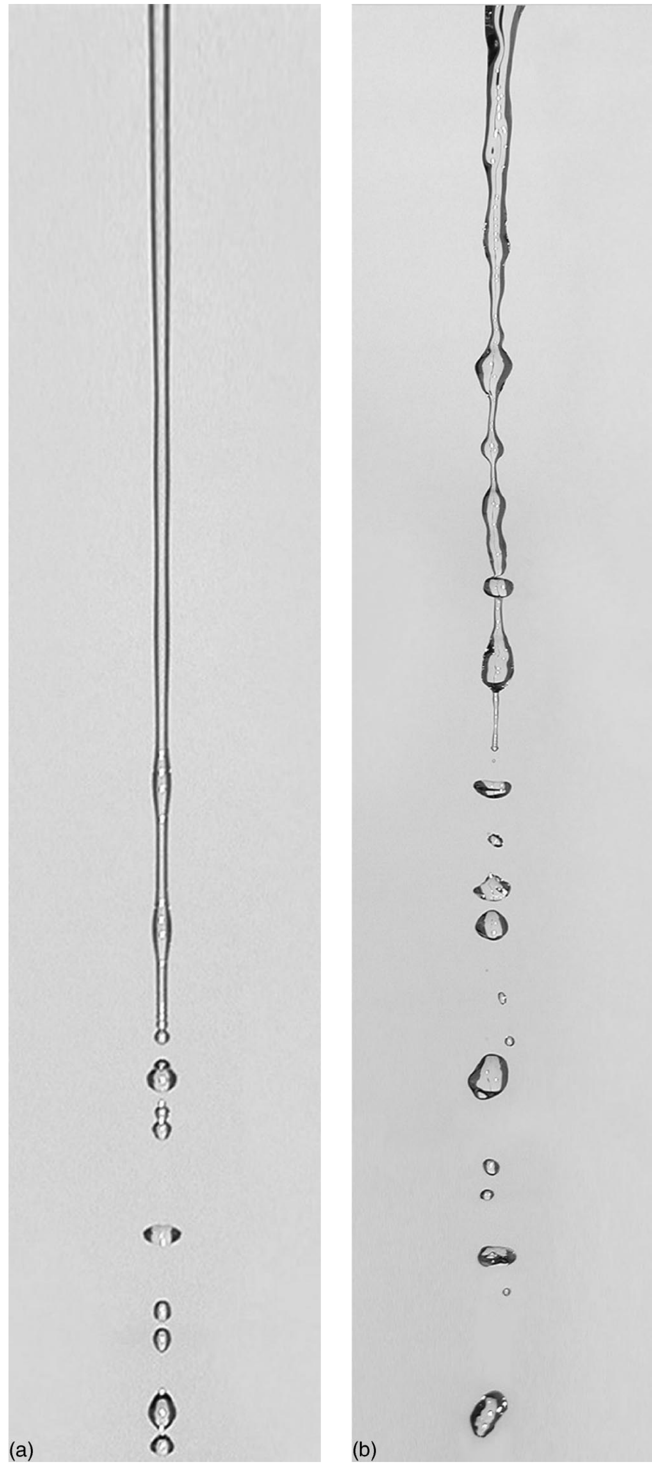

Fig. 8. Drop formation showing the dependence on the irregularity of a stream of water: (a) no external forcing and (b) additional external perturbations.

where $R_{2,\mathrm{I}}$ and $R_{2,\mathrm{II}}$ are the principal radii of the surface curvature.

At the beginning of drop formation, where the stream swells slightly, Eq. (9) is a good approximation for the driving force because $R_{2,\mathrm{II}} \gg R_{2,\mathrm{I}}$ and Eq. (10) simplifies to Eq. (8). However, in the later process of drop growth, $R_{2,\mathrm{I}}$ increases and $R_{2,\mathrm{II}}$ decreases. Finally, when the drop is close to its spherical form (see Fig. 6), $R_{2,\mathrm{I}} = R_{2,\mathrm{II}} = R_2$, and the pressure in the drop takes the value

$$p_2 - p_0 = \frac{2\gamma}{R_2}. \tag{11}$$

Equation (11) also can be obtained directly by reasoning similar to that leading to Eq. (6). The difference between the pressure in the drop, p_2, and the surrounding air pressure, $p_2 - p_0$, acting on the surface of the sphere equals the surface tension $F = \gamma 2\pi R_2$, which gives $(p_2 - p_0)\pi R_2^2 = \gamma 2\pi R_2$, and hence Eq. (11) follows.

The driving force for the liquid flowing into the spherical drop is given by

$$p_1 - p_2 = \gamma\left(\frac{1}{R_1} - \frac{2}{R_2}\right). \tag{12}$$

During the growth of a drop, R_1 becomes smaller and R_2 larger. According to Eq. (12), the driving force for the flow of liquid into the drop becomes even larger. The liquid is sucked into the drop, and, finally, the drop is separated from the stream.

The dynamics of drop formation depends crucially on the type of fluid. For a more comprehensive treatment of the dynamics, see Ref. 19. It is shown that water does not produce long necks, but that oil can.[16]

IV. DISCUSSION

Because of gravity, a stream of liquid becomes narrower as it falls. At the bottom of the stream, it swells again and drops are formed. Here we discuss the influence of the surface tension and external perturbations on the decay of a liquid stream.

According to Eqs. (9) and (12), a larger surface tension causes a larger driving force for the influx of liquid into drops, and drop formation is accelerated; hence, the drops are formed earlier in the stream, that is, closer to the nozzle. The influence of surface tension on drop formation can be tested by comparing drop formation in a stream of fresh water and in a stream of soapy water (see Fig. 7), which has a smaller surface tension than fresh water. We see that drops are formed much earlier in a fresh water stream ($h = 5 \pm 1$ cm) than in a soapy water stream ($h = 15 \pm 2$ cm). Both streams have the same initial cross-section S_0 and velocity v_0.

Another important issue is the influence of perturbations on the decay of a liquid stream, which can be simply demonstrated by a slight shaking of the nozzle. If the external perturbation is larger, measurable swellings in the stream appear earlier, and therefore the swellings also grow earlier into drops, that is, the drops appear closer to the nozzle (Fig. 8). A more comprehensive discussion of the influence of perturbations on the decay of liquid streams is given in Ref. 8.

More viscous liquids can be studied experimentally. However, for highly viscous liquids, viscous dissipation can be more important than inertia as the primary resistance against surface tension, and, therefore, drop formation is mostly the result of the balance between surface tension and viscous dissipation. In general, a combination of all these effects, surface tension, inertia, and viscous dissipation, is important for drop formation in viscous liquids.

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