FINITE SPHERICAL WELL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.9.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.6.9.

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A variant on the infinite spherical well is the finite spherical well, with potential

$$V(r)=\begin{cases}-V_0 & r<a\\ 0 & r>a\end{cases} \tag{1}$$

This problem is superficially like that of the finite square well in one dimension, but there is a crucial difference, which is that the variable r starts at 0 rather than $-a$, so we can't use the argument that the wave function is even or odd. However, we have worked out a similar one-dimensional problem with the hybrid square well, and we can adapt that solution to this problem.

The wave function must be found in the two regions separately, and then boundary conditions used to determine the energies.

For $r<a$, the radial equation is (with $l=0$)

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}-V_0u = Eu \tag{2}$$

$$\frac{d^2u}{dr^2} = -\frac{2m}{\hbar^2}(V_0+E)u \tag{3}$$

$$\equiv -\mu^2u \tag{4}$$

where

$$\mu=\sqrt{\frac{2m(V_0+E)}{\hbar^2}} \tag{5}$$

This has the general solution

$$u(r) = C\sin\mu r + D\cos\mu r \tag{6}$$

As with the infinite square well, we note that the actual radial function is $u(r)/r$, so we must eliminate the cosine term to keep the radial function finite at $r = 0$. Therefore

$$u(r) = C\sin\mu r \tag{7}$$

For $r > a$, the equation is

$$\frac{d^2u}{dr^2} = k^2u \tag{8}$$

where

$$k \equiv \sqrt{-\frac{2mE}{\hbar^2}} \tag{9}$$

Note that for a bound state, E is negative, so k is real. This equation has a general solution

$$u(r) = Ae^{kr} + Be^{-kr} \tag{10}$$

and in order for the function to remain finite at infinity, we must set $A = 0$ so we have:

$$u(r) = Be^{-kr} \tag{11}$$

Now for the boundary conditions. We have only one boundary, at $r = a$, so as with the square well in the one-dimensional case, we require the function and its first derivative to be continuous at the boundary. These conditions give us

$$C\sin\mu a = Be^{-ka} \tag{12}$$
$$\mu C\cos\mu a = -kBe^{-ka} \tag{13}$$

Eliminating the exponential by dividing these two equations gives us a condition similar to that in the square well case:

$$-\frac{\mu}{k} = \tan\mu a \tag{14}$$

We can look for solutions graphically. As before we introduce two variables

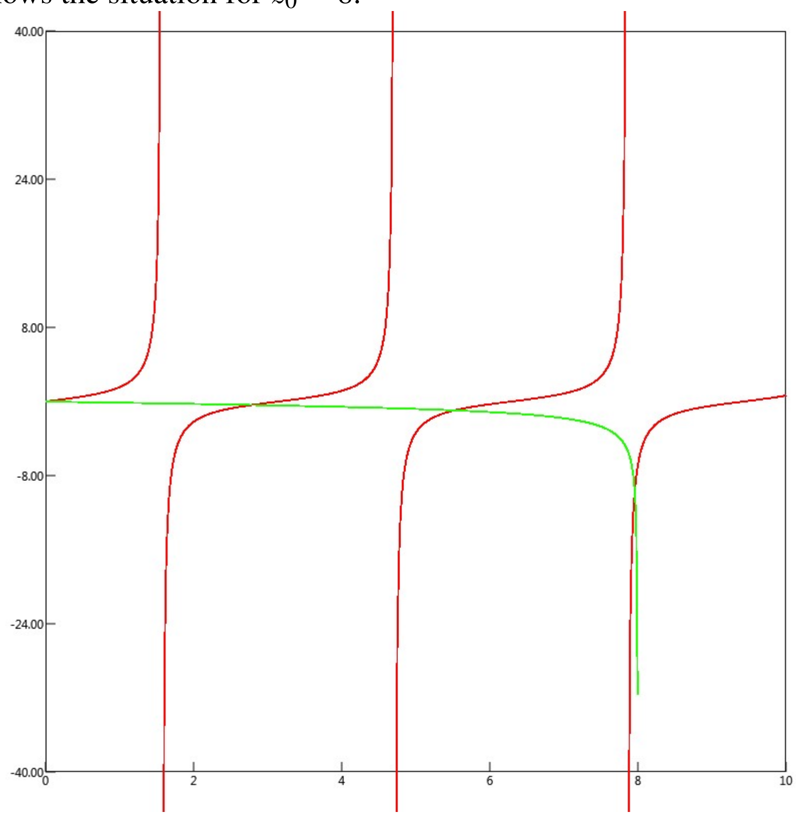
$$z \equiv \mu a \tag{15}$$

$$z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0} \tag{16}$$

Then $ka = \sqrt{z_0^2 - z^2}$ and the equation to solve is

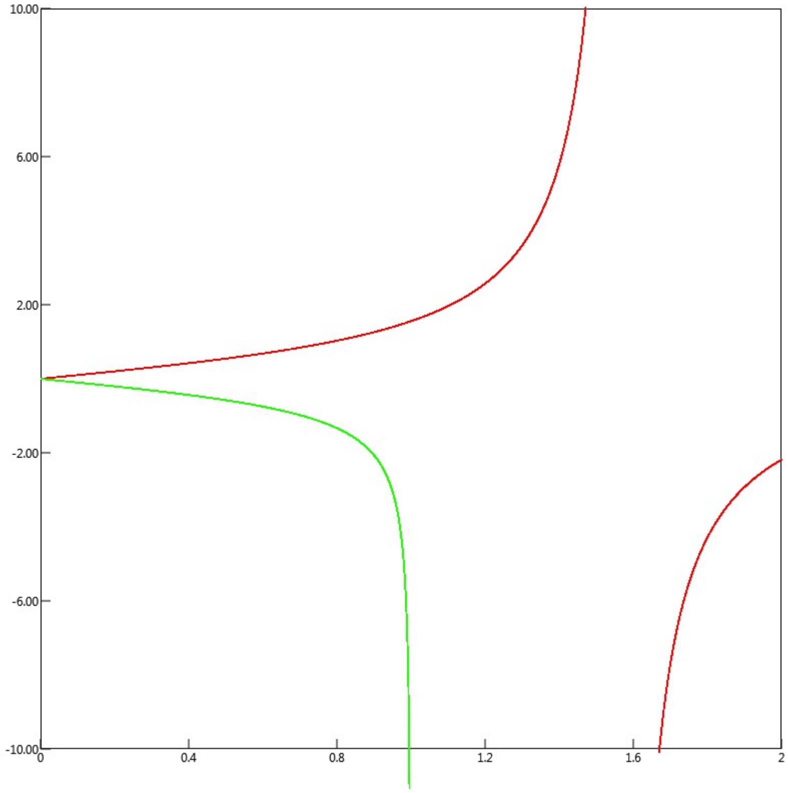
$$\tan z = \frac{-1}{\sqrt{z_0^2/z^2 - 1}} \tag{17}$$

The number of solutions depends on the value chosen for z_0. The plot shows the situation for $z_0 = 8$.



The green curve is $\frac{-1}{\sqrt{z_0^2/z^2-1}}$ and the red curve is $\tan z$. In this case we can see there are 3 intersections, so here there are 3 bound states. The precise values of the energies can be found by solving the equation numerically using software such as Maple's fsolve command.

The asymptote of $\frac{-1}{\sqrt{z_0^2/z^2-1}}$ is at $z = z_0$ so since the first asymptote for the tangent is at $z = \pi/2$, clearly if $z_0 < \pi/2$ the two curves will not intersect. The following plot shows the situation for $z_0 = 1$:



The condition $z_0 < \pi/2$ translates into

$$V_0 a^2 < \frac{\hbar^2\pi^2}{8m} \tag{18}$$

Incidentally, the intersection at $z = 0$ isn't a physical solution, since it implies $E = -V_0$, which in turn means $d^2u/dr^2 = 0$ and $\mu = 0$, giving $u = Cr + D$, $R = u/r = C + D/r$. To avoid infinity at the origin, we must have $B = 0$, and to satisfy continuity of the wave function and its derivative at $r = a$ (see above) gives $C = Be^{-ka}$ for the wave function and $-kBe^{-ka} = 0$ for its derivative. The latter condition means $B = 0$ and hence $C = 0$, meaning the wave function is zero everywhere and not normalizable.