FORMULACION COVARIANTE DE LA ELECTRODINAMICA

(EXPRESIONES MANIFIESTAMENTE COVARIANTES DE MAXWELL)

i) TETRAVECTOR DENSIDAD DE CORRIENTE

DEF: • $J_0 = c\rho$ • $\vec{J} \equiv (J^1, J^2, J^3)$

⇒ EQUACION DE CONTINUIDAD

• $$\sum_{\mu=0}^{3} \frac{\partial J^\mu}{\partial x^\mu} = 0$$

(DIVERGENCIA DE TETRAVECTOR IGUALADA AL TENSOR 0)

• $$\sum_{\mu=0}^{3} \frac{\partial A^\mu(x^\beta)}{\partial x^\mu} = \frac{\partial A^1}{\partial x^1} + \frac{\partial A^2}{\partial x^2} + \frac{\partial A^3}{\partial x^3} - \frac{\partial A^0}{\partial x^0}$$ ESCALAR!

N partículas

$x^0 = ct \Rightarrow$

$$\bullet\ J^\mu(\vec{x},t) = \sum_{n=1}^{N} q_n\, \delta^3(\vec{x}-\vec{x}_n(t))\, \frac{dx^\mu_n(t)}{dt}$$

(SATISFACE CONTINUIDAD)

$$* \ J^\mu(\vec{x},t) = \int dt' \sum_{n=1}^{N} q_n\, \delta^4(\vec{x}-\vec{x}_n(t))\, \frac{dx^\mu_n(t)}{dt'}$$

dt' SE CANCELA $\rightarrow$ $dt' \rightarrow d\tau \Rightarrow$ (τ : TIEMPO PROPIO)

$$\bullet\ J^\mu(\vec{x},t) = \int d\tau \sum_{n=1}^{N} q_n\, \underbrace{\delta^4(x-x_n(\tau))}_{\text{(ESCALAR LORENTZ)}}\, \frac{\overbrace{dx^\mu_n(\tau)}^{\text{(TETRAVECTOR)}}}{d\tau}$$

$\Downarrow$

$J^\mu(\vec{x},t)$ SE COMPORTA TETRAVECTOR SI

LA CARGA ELECTRICA ES INVARIANTE

* CARGA TOTAL

$$Q = \int d^3x\, j^0(x)$$

$$\Rightarrow \quad \frac{dQ}{dt} = c \int d^3x\, \frac{\partial j^0(x)}{\partial x^0}$$

$\Downarrow$ EC. CONTINUIDAD

$$\frac{dQ}{dt} = c \int d^3x\, \vec{\nabla} \cdot \vec{j}(\vec{x}) = 0$$

(GAUSS)

(SI $\vec{j}(x) \to 0$ CUANDO $x \to \infty$)

- SE CONSERVA

* CARGA Q ES ESCALAR DE LORENTZ

- $\theta(x)$: HEAVISIDE
- $\dfrac{d\theta(x)}{dx} = \delta(x)$
- n_β: TIPO TIEMPO ($n_0 = -1$; $n_1 = n_2 = n_3 = 0$)

$\Downarrow$

- $$Q = \int d^3x\, J^0(x) = \int d^4x\, J^\mu(x) \frac{\partial \theta(n_\beta x^\beta)}{\partial x^\mu}$$

Q TRANSFORMADA LORENTZ

- $$Q' = \int d^4x\, J^\mu(x) \frac{\partial \theta(n'_\beta x^\beta)}{\partial x^\mu}$$
- $n'_\beta = L^\alpha_\beta\, n_\alpha$

$\Downarrow$ (EC. CONTINUIDAD)

- $$Q' - Q = \int d^4x \frac{\partial}{\partial x^\mu}\left\{ J^\mu(x)\left[\theta(n'_\beta x^\beta) - \theta(n_\beta x^\beta)\right]\right\}$$

$\Downarrow$ (GAUSS)

$$Q' - Q = \int d\sigma_\mu\, J^\mu(x)\left[\theta(n'_\beta x^\beta) - \theta(n_\beta x^\beta)\right]$$

$d\sigma_\mu = \{dx_0\,dx_2\,dx_3,\ dx_0\,dx_1\,dx_3,\ dx_0\,dx_1\,dx_2,\ dx_1\,dx_2\,dx_3\}$

(SUPERFICIE 3D EN 4D)

n'_β : TETRAVECTOR TIPO TIEMPO $\Rightarrow$

$$\left[\theta(n'_\beta x^\beta) - \theta(n_\beta x^\beta)\right] \xrightarrow[t \to \infty]{} 0$$

(X FIJO)

Y: $J^\mu(x) \xrightarrow[x \to \infty]{} 0$

(t FIJO)

SI SUPERFICIE $\to \infty$

$$Q' - Q = 0$$

(ESCALAR DE LORENTZ)

EVIDENCIA EXPERIMENTAL DE LA INDEPENDENCIA DE Q CON EL ESTADO DE MOVIMIENTO

LOS ATOMOS SON NEUTROS !

(ELECTRONES SE MUEVEN)

* TENSOR DE CAMPO ELECTROMAGNETICO

ESCRIBIR MAXWELL EN FORMA COVARIANTE

(NO SIMPLE CON LOS TRIVECTORES $\vec{E}$ Y $\vec{B}$) $(\vec{E}(3), \vec{B}(3))$

UNIFICAR EL TRATAMIENTO CON UN

CAMPO TENSORIAL DE $r=2$. ANTISIMETRICO

(6 CANTIDADES INDEPENDIENTES)

* MAXWELL INHOMOGENEAS:

- $k''c\left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}\right) - \frac{\partial E_1}{\partial x_0} = \frac{4\pi k}{\epsilon_0 c} J_1$

- $k''c\left(-\frac{\partial B_3}{\partial x_2} + \frac{\partial B_1}{\partial x_3}\right) - \frac{\partial E_2}{\partial x_0} = \frac{4\pi k}{\epsilon_0 c} J_2$

- $k''c\left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2}\right) - \frac{\partial E_3}{\partial x_0} = \frac{4\pi k}{\epsilon_0 c} J_3$

- $\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} = \frac{4\pi k}{\epsilon_0 c} J_0$

* TENSOR DE CAMPO ELECTROMAGNETICO $F_{\mu\nu}$

- $$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & k''cB_3 & -k''cB_2 \\ -E_2 & -k''cB_3 & 0 & k''cB_1 \\ -E_3 & k''cB_2 & -k''cB_1 & 0 \end{pmatrix}$$

$\downarrow$

MAXWELL:

- $$\sum_{\nu=0}^{3} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi k}{\epsilon_0 c} J_\mu$$

- $$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

TETRADIVERGENCIA DE TENSOR DE $r=2$ ES UN TETRAVECTOR

$\Rightarrow F_{\mu\nu}$: VERDADERO CAMPO TENSORIAL !.

CONSISTENCIA CONTINUIDAD:

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\underbrace{\frac{\partial^2 F_{\mu\nu}}{\partial x_\mu \partial x_\nu}}_{=0} = \frac{4\pi K}{\epsilon_0 c}\sum_{\mu=0}^{3}\frac{\partial J_\mu}{\partial x_\mu} = 0$$

(CONTRACCION S y A)

* MAXWELL HOMOGENEAS:

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\wedge E = -K'' \dfrac{\partial\vec{B}}{\partial t} \end{cases}$$

↓

$$\frac{\partial F_{\mu\nu}}{\partial x_\sigma} + \frac{\partial F_{\sigma\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\sigma}}{\partial x_\mu} = 0$$

(4 ECUACIONES (0,1,2,3))

INDICES DIFERENTES!   (2 IGUALES ⇒ IDENTIDAD TRIVIAL)

$$\left(\underline{EJ}: \ \frac{\partial F_{\sigma\nu}}{\partial x_\nu} + \frac{\partial F_{\nu\sigma}}{\partial x_\nu} = 0 \right)$$

* COVARIANCIA:

INTRODUCIR TENSOR DUAL A $F_{\mu\nu}$

$$\bullet\quad F^{D}_{\mu\nu} = \sum_{\beta=0}^{3}\sum_{\alpha=0}^{3} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

(LEVI - CIVITA)

$$\frac{\partial F_{\mu\nu}}{\partial x_\sigma} + \frac{\partial F_{\sigma\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\sigma}}{\partial x_\mu} = 0$$

$$\Downarrow$$

$$\bullet\quad \sum_{\nu=0}^{3} \frac{\partial F^{D}_{\mu\nu}}{\partial x_\nu} = 0$$

TETRADIVERGENCIA DEL DUAL $\Rightarrow$ EXPLICITAMENTE COVARIANTE

ó

$$\sum_{\beta=0}^{3}\sum_{\alpha=0}^{3}\sum_{\nu=0}^{3} \epsilon^{\mu\nu\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial x_\nu} = 0$$

DUAL

0 $\Rightarrow$ NO FUENTES !

* TRANSFORMACIONES DE LORENTZ DE LOS CAMPOS

$$\bullet\ F'_{\mu\nu} = b^{\beta}_{\mu}\, b^{\alpha}_{\nu}\, F_{\beta\alpha}$$

MATRICIAL: $F' = B F B^T$

(S y S' con v a lo largo de x_1)

$$B = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\rightarrow$

- $E'_1 = E_1$
- $E'_2 = \gamma\left(E_2 - k'' c\beta B_3\right)$
- $E'_3 = \gamma\left(E_3 + k'' c\beta B_2\right)$
- $B'_1 = B_1$
- $B'_2 = \gamma\left(B_2 + \frac{1}{k'' c}\beta E_3\right)$
- $B'_3 = \gamma\left(B_3 - \frac{1}{k'' c}\beta E_2\right)$

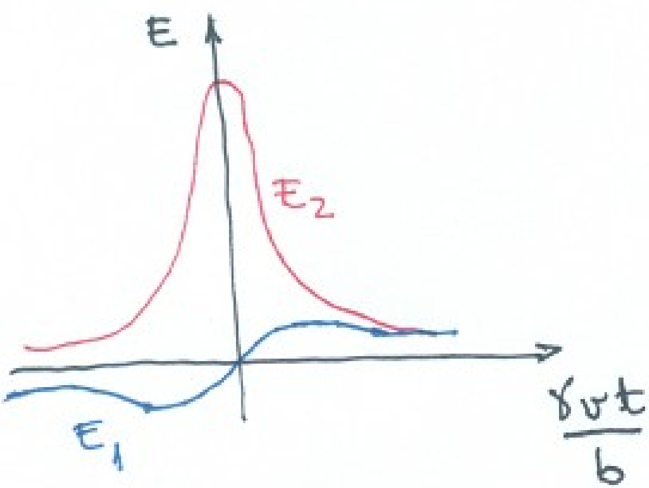
NOTA: (EJ:) EN S SOLO UN CAMPO ($\vec{E}$ ó $\vec{B}$)
EN S' CONTRIBUYEN AMBOS

EJ: CARGA PUNTUAL EN MOVIMIENTO (RECTILINEO Y v = CTE.)

$$\vec{E}' = \frac{k}{\epsilon_0}\frac{q}{r'^2}\hat{r}' \quad ; \quad \vec{B}' = 0 \qquad (\vec{r}' \text{ ubica } P \text{ en } S')$$

$$\begin{cases} t' = \gamma t \\ x' = -\gamma v t \\ y' = b \\ z' = 0 \end{cases} \Rightarrow$$

- $E_1 = -\frac{k}{\epsilon_0}\frac{q\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$
- $E_2 = \frac{k}{\epsilon_0}\frac{q\gamma b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$
- $E_3 = 0$
- $B_1 = B_2 = 0$
- $B_3 = \frac{k}{k'' \epsilon_0 c}\frac{q\gamma\beta b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$



(DISTANCIA GRANDE: $\frac{x}{b} \gg 1$)

* PROPIEDADES BAJO $\underline{\underline{P}}$ Y $\underline{T}$

- COMPONENTES DE $\vec{E}$ $\Rightarrow$ INDICES μ ó ν TEMPORAL DE $F_{\mu\nu}$
- COMPONENTES DE $\vec{B}$ $\Rightarrow$ INDICES ESPACIALES

EN TENSOR DUAL $F^{D}_{\mu\nu}$ SE INTERCAMBIA

$\vec{E}$: VECTOR POLAR

$\vec{B}$: VECTOR AXIAL (x)

- $E_i \xrightarrow[P]{} E'_i = -E_i$ (POLAR)

- $B_i \xrightarrow[P]{} B'_i = B_i$ (AXIAL)

- MAXWELL INVARIANTE P SI • $\begin{cases} \rho \xrightarrow[P]{} \rho' = \rho \\ J_i \xrightarrow[P]{} J'_i = -J_i \end{cases}$

* $\underline{\underline{T}}$:

- $B_i \xrightarrow[T]{} B'_i = -B_i$
- $E_i \xrightarrow[T]{} E'_i = E_i$

* POTENCIAL ELECTROMAGNETICO

- $$F_{\beta\gamma} = \frac{\partial A_\gamma}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\gamma}$$

- $\underline{A_\gamma}$: TETRAVECTOR POTENCIAL ELECTROMAGNETICO

$\downarrow$ (MAXWELL)

$$\sum_{\nu=0}^{3} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \sum_{\nu=0}^{3} \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right)$$

$$= \frac{\partial}{\partial x^\mu}\underbrace{\left(\sum_{\nu=0}^{3} \frac{\partial A_\nu}{\partial x_\nu}\right)}_{\text{TETRADIVERGENCIA}} - \underbrace{\left(\sum_{\nu=0}^{3} \frac{\partial^2}{\partial x_\nu \partial x_\mu}\right)}_{\text{D'ALAMBERTIANO}} A_\mu = \frac{4\pi K}{\varepsilon_0 c} J_\mu$$

INVARIANCIA DE GAUGE $\Rightarrow$ $A_\beta \to A_\beta + \dfrac{\partial \phi}{\partial x_\beta}$

(GAUGE DE LORENTZ $\displaystyle\sum_{\nu=0}^{3} \frac{\partial A_\nu}{\partial x_\nu} = 0$)

$\downarrow$

$$\square^2 A_\mu = -\frac{4\pi K}{\epsilon_0 c} J_\mu$$

GAUGE DE LORENTZ: COVARIANTE LORENTZ !

$J_\mu = 0 \Rightarrow \square^2 A_\mu = 0 \Rightarrow A_\mu = C_\mu e^{ik.x}$ (C_μ: POLARIZACION)

FASE ESCALAR $\Rightarrow$ k_ρ TETRAVECTOR CON $k_0 = \dfrac{\omega}{c}$; $k_\rho k^\rho = 0$