A MECHANISM OF MAGNETIC HYSTERESIS IN HETEROGENEOUS ALLOYS

By E. C. STONER, F.R.S. and E. P. WOHLFARTH
Physics Department, University of Leeds

(*Received* 24 *July* 1947)

The Becker-Kersten treatment of domain boundary movements is widely applicable in the interpretation of magnetization curves, but it does not account satisfactorily for the higher coercivities obtained, for example, in permanent magnet alloys. It is suggested that in many ferromagnetic materials there may occur 'particles' (this term including atomic segregates or 'islands' in alloys), distinct in magnetic character from the general matrix, and below the critical size, depending on shape, for which domain boundary formation is energetically possible. For such single-domain particles, change of magnetization can take place only by rotation of the magnetization vector, I_0. As the field changes continuously, the resolved magnetization, I_H, may change discontinuously at critical values, H_0, of the field. The character of the magnetization curves depends on the degree of magnetic anisotropy of the particle, and on the orientation of 'easy axes' with respect to the field. The magnetic anisotropy may arise from the shape of the particle, from magneto-crystalline effects, and from strain.

A detailed quantitative treatment is given of the effect of shape anisotropy when the particles have the form of ellipsoids of revolution (§§ 2, 3, 4), and a less detailed treatment for the general ellipsoidal form (§ 5). For the first it is convenient to use the non-dimensional parameter h, such that $h = H/(|N_a - N_b|)\, I_0$, N_a and N_b being the demagnetization coefficients along the polar and equatorial axes. The results are presented in tables and diagrams giving the variation with h of I_H/I_0. For the special limiting form of the oblate spheroid there is no hysteresis. For the prolate spheroid, as the orientation angle, θ, varies from 0 to 90°, the cyclic magnetization curves change from a rectangular form with $|h_0| = 1$, to a linear non-hysteretic form, with an interesting sequence of intermediate forms. Exact expressions are obtained for the dependence of h_0 on θ, and curves for random distribution are computed.

All the numerical results are applicable when the anisotropy is due to longitudinal stress, when $h = HI_0/3\lambda\sigma$, where λ is the saturation magnetostriction coefficient, and σ the stress. The results also apply to magneto-crystalline anisotropy in the important and representative case in which there is a unique axis of easy magnetization as for hexagonal cobalt. Estimates are made of the magnitude of the effect of the various types of anisotropy. For iron the maximum coercivities, for the most favourable orientation, due to the magneto-crystalline and strain effects are about 400 and 600 respectively. These values are exceeded by those due to the shape effect in prolate spheroids if the dimensional ratio, m, is greater than 1·1; for $m = 10$, the corresponding value would be about 10,000 (§ 7).

A fairly precise estimate is made of the lower limit for the equatorial diameter of a particle in the form of a prolate spheroid below which boundary formation cannot occur. As m varies from 1 (the sphere) to 10, this varies from 1·5 to $6{\cdot}1 \times 10^{-6}$ for iron, and from 6·2 to 25×10^{-6} for nickel (§ 6).

A discussion is given (§ 7) of the application of these results to (*a*) non-ferromagnetic metals and alloys containing ferromagnetic 'impurities', (*b*) powder magnets, (*c*) high coercivity alloys of the dispersion hardening type. In connexion with (*c*) the possible bearing on the effects of cooling in a magnetic field is indicated.

1. Introduction

Since about 1930 considerable progress has been made in the understanding of the behaviour of ferromagnetics in low and moderate fields. This is mainly due to the development of an interpretative scheme, to which many have contributed from both the experimental and

theoretical sides, in which change of magnetization is associated with movement of boundary walls between regions, or domains, in which the magnetization is in different directions. In each domain the absolute value of the intensity of magnetization remains equal to the quasi-saturation value appropriate to the temperature. In this paper, consideration is given to an entirely different mechanism of change of magnetization which may be physically relevant to the behaviour of heterogeneous alloys in which a more strongly ferromagnetic phase is finely dispersed in a less strongly ferromagnetic matrix. The particular interest of this mechanism is that it can account, in a simple and unforced way, for very high coercivity values. The process envisaged, even in alloys in which it may be predominant, will usually be accompanied by processes involving the movement of domain boundaries, and in those alloys in which it cannot be of major importance, it may none the less be contributory to the magnetic behaviour. For these reasons it is appropriate to indicate the essentials of the theory of the boundary movement process, and to draw attention to its limitations. This is perhaps the more desirable in that it seems to be widely supposed that the behaviour of ferromagnetics in low and moderate fields, and in particular hysteresis effects, can, in principle, be explained exclusively in terms of the boundary movement process, while at the same time there is often misunderstanding about some of the necessary assumptions in the theoretical treatment. Many of the leading ideas are due to Becker, who has given, with Döring, a comprehensive account of the general theory of 'technical' magnetization curves in the book *Ferromagnetismus* (1939); fuller details, and references to the earlier original papers, may be found in this book.

(i) *The boundary movement process*

Boundary energy. The energy associated with a domain boundary is dependent jointly on the interchange interaction and the magneto-crystalline, or magneto-elastic anisotropy in the region where the boundary occurs. If the effective width of the boundary is such as to minimize the energy, it may be shown that the energy per unit area, γ, is given by

$$\gamma = c_1 a(JC)^{\frac{1}{2}}, \tag{1·1}$$

where a is the lattice constant, J a measure of the interchange interaction energy per unit volume, C, with the dimensions of energy density, or stress, a measure of the anisotropy, and c_1 a numerical factor, appropriate to the precise conditions (depending, for example, on the angle, most commonly 90° or 180°, between the directions of magnetization on the two sides of the domain boundary) and depending on the manner in which J and C are specified. The calculation of C in the general case would be very difficult, but a simple representative expression is obtainable for a specimen subjected to a uniform tension, σ, of a material whose (saturation) magnetostriction coefficient, λ, is independent of direction and whose magneto-crystalline anisotropy is negligibly small. In this case, the order of magnitude of C is given by

$$C = \lambda\sigma. \tag{1·2}$$

For cubic crystals, the dependence of the magneto-crystalline energy, E_c, on the direction of magnetization is given approximately by

$$E_c = K(\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2), \tag{1·3}$$

where α_1, α_2, α_3 are the direction cosines of I_0 with respect to the cubic axes. If the strain anisotropy is negligible, the order of magnitude of C is given by

$$C = K', \tag{1·4}$$

where K' is the difference between the maximum and minimum values of E_c (e.g. $K' = K/3$ with the expression (1·3)). The forms (1·2) and (1·4) are appropriate for $\lambda\sigma \gg K$, and $\lambda\sigma \ll K$, respectively. (As to orders of magnitude it may be noted that for nickel and iron the magneto-elastic effect becomes comparable with the magneto-crystalline effect for values of σ of about 4 and 80 kg.mm.$^{-2}$ respectively.)

Effect of stress variations. Whether or not the strain energy predominates over the crystal energy, the variation of the boundary energy per unit area as it moves parallel to itself will depend primarily on the variation of internal stress, having the character of a tension or compression, in the material. It is mainly on this basis that the theory of domain boundary movement has been developed, particularly, in the present connexion, by Kersten (1938), following on earlier work by Kondorsky (1937). In the absence of a field a boundary will occupy a position for which the energy, γ, is a minimum. In an applied field the boundary will move to a new equilibrium position for which the sum of the field energy ($-\mathbf{H}\cdot\mathbf{I}_0$ per unit volume) and the boundary energy is a minimum. If, for example, the central plane of the boundary is in the yz plane at $x = 0$, and, outside the boundary region, I_0 is parallel to the field for x positive, and antiparallel for x negative, the equilibrium position will be such that

$$2HI_0 = d\gamma/dx. \tag{1·5}$$

As H increases the change in magnetization associated with the boundary movement will be reversible until the field attains a value H_0 for which the boundary reaches a position at which the energy gradient is a maximum. At this value, namely

$$H_0 = \tfrac{1}{2}(d\gamma/dx)_{\max.}/I_0, \tag{1·6}$$

the boundary will move spontaneously (i.e. without further increase of field) and irreversibly to a new position of equilibrium, the process corresponding to a Barkhausen jump. In many cases it is reasonable to associate the position variation of γ with the presence of localized internal stress, and to relate $(d\gamma/dx)_{\max.}$ with the amplitude and form of the internal stress variations. Kersten has considered the effect of various types of stress distribution, and obtains the result

$$H_0 = p_0(\lambda\sigma_i/I_0), \tag{1·7}$$

where σ_i is the internal stress amplitude (the difference from the mean stress, and so effectively the value of the internal stress where it has a maximum, when there is no applied stress), and p_0 a numerical factor depending on the distribution. Denoting the effective width of a transition zone by δ, and the effective width of the stress 'hump' by l, Kersten finds that p_0 is proportional to δ/l for $\delta \ll l$, and to l/δ for $\delta \gg l$, and that for $\delta \sim l$, p_0 reaches its maximum value of order unity.

The coercivity, H_c, is an appropriate average of values of H_0 throughout the material, and cannot be estimated in any precise way. The maximum value of the coercivity can, however, be estimated approximately as

$$(H_c)_{\max.} \doteqdot \tfrac{3}{2}(\lambda\sigma_i/I_0). \tag{1·8}$$

By plausible and not too specialized assumptions about the internal stresses in ferromagnetics under various conditions, as dependent on thermal and mechanical treatment and on the presence of heterogeneously dispersed foreign materials, a comprehensive qualitative co-ordination can be made of such magnetic characteristics as initial permeability and coercivity for a wide variety of materials.

Boundary area effects. In the treatment, outlined above, of the effect of stress variations, it has usually been tacitly assumed that the movement of a domain boundary is not accompanied by any appreciable change in its surface area. Any such change, however, is necessarily accompanied by a change in boundary energy, even in the absence of stress inhomogeneities. This effect has recently been considered by Kersten (1943 *a*), who concludes that it may be of predominant importance in ferromagnetics containing relatively small proportions of foreign, non-ferromagnetic, bodies, or impurities, as do many of the technically important ferromagnetic materials. A boundary between domains of the ferromagnetic matrix may pass through, or include within itself, impurity particles, which do not contribute to the boundary energy. For a given gross area of a boundary, the energy will be a maximum when no impurity particles are included. By idealizing the impurity particles as spherical and uniformly distributed, estimates can be made of the increase of boundary energy in the movement of a boundary from a zero field equilibrium position, in which the central plane of the boundary passes through a maximum number of impurity particles, and expressions obtained for both initial permeability and coercivity, in terms of the concentration of the impurity, and the ratio of the mean particle diameter, d, to the effective thickness, δ, of the boundary wall. For $d \gg \delta$, Kersten obtains, among other relations, as an approximate expression for the coercivity,

$$H_c = c_2(K'/I_0)\,(\delta/d)\,\alpha^{\frac{2}{3}}, \tag{1·9}$$

where $c_2 = 6^{\frac{2}{3}}\pi^{\frac{1}{3}} \doteqdot 5$, and α is the volume concentration of the impurity. For $d \ll \delta$, (δ/d) in (1·9) is replaced by $(d/2\delta)$. It is shown by Kersten that the experimental results on annealed carbon iron are covered very satisfactorily by (1·9) with values for d which are in reasonable agreement with those estimated for the cementite particles from metallographical studies. A fuller discussion of the application of the treatment to the interpretation of the magnetic properties of various ferromagnetic materials is given in a doctorate dissertation (Kersten 1943 *b*).

Limitations of the boundary movement treatment. The most direct evidence for the effect of stress on boundary movement, and indeed, for the physical occurrence of the boundary movement process, is derived from experiments on large Barkhausen discontinuities in specimens (wires) under uniform tension. The quantitative applicability of the basic ideas to the interpretation of the effect of uniform stress on magnetization curves has been abundantly confirmed. Uncertainties arise, however, in the extension of these ideas to the explanation of the low and moderate field behaviour of (unstressed) materials generally in terms of localized internal stress variations. By postulating a particular type of variation, a qualitative co-ordination of various magnetic characteristics of a particular material can, indeed, be obtained; but there is, in general, no independent evidence that the actual internal stress variations are, in fact, of the type required to account for the magnetic behaviour. It cannot be too strongly emphasized that the variations required are not of the type corre-

sponding to local fluctuations of density, such as might be associated with variations of stress having the character of 'hydrostatic' pressure, but are variations in spacing such as would arise from variations in tension along a particular direction.

For well-annealed materials containing small amounts of 'impurities' the interpretation of the magnetic properties in terms of changes in the effective area of domain boundaries, as suggested by Kersten in his later work, seems very convincing, and the stress variation mechanism, here highly improbable, need not be invoked. As Kersten points out, this latter mechanism would be expected to be of importance primarily for cold worked materials, to which the general theory has been applied with the greatest success. It may be noted that the magnitude of the stress (that is of σ_i in (1·8) and in related expressions for initial permeability) required to account for the magnetic characteristics of such materials is of the order 10 kg.mm.$^{-2}$.

For materials of very high coercivity (say $H_c \geqslant 500$), such as the newer permanent magnet alloys, the stress variation mechanism might at first sight seem very plausible, for many of these materials are hard and brittle, characteristics which may be associated with a state of high localized internal strain. The minimum value of the internal stress required to account for the observed coercivity as estimated from (1·8), is, however, in some cases about 200 kg.mm.$^{-2}$, which is of the same order as the breaking stress. Although such high values may not be 'impossible' the assumption of variations in internal stress (tension) of this magnitude over distances of a few tens of atoms must at present be regarded as somewhat speculative. Moreover, some of the high coercivity alloys are malleable and ductile, which is hardly compatible with large internal stress variations. It seems, therefore, very improbable that the magnetic behaviour of materials of very high coercivity is to be accounted for solely in terms of the effect of internal stress variations on the movement of domain boundaries. The boundary area, or 'foreign body' effect, may be of major importance in some of the older magnet alloys with coercivities of the order of 50 (such as tungsten and chromium steels), but it can hardly account for coercivities more than ten times as great, as becomes clear on inserting the most favourable numerical values in (1·9).

In alloys containing only small amounts of ferromagnetic material, in some cases simply as a ferromagnetic 'impurity', a relatively high coercivity is sometimes found. (More detailed reference to the experimental results is made in § 7.) For materials of this type a general explanation of the magnetic properties cannot be given in terms of boundary movements, for the volume of the particles of the dispersed ferromagnetic phase may be too small for the formation of a domain boundary to be energetically possible.

Although, then, the hysteretic properties of ferromagnetic materials are probably to be accounted for mainly in terms of factors affecting domain boundary movement, the examples just mentioned suggest that processes other than boundary movement may occur, and in some cases be of predominant importance. One such process is considered in detail in this paper.

(ii) *The rotation process in single domains*

In ordinary materials the boundary movement process, by itself, would bring the specimen to a state in which the magnetization in each region is directed along one of the easiest directions of magnetization (as determined by the state of strain and crystal orientation in the region) in a sense which gives the maximum resultant magnetization in the field direction.

It is accompanied by, and followed in higher fields by, a process in which the magnetization vector for the region rotates from the easy direction towards the field direction. In a heterogeneous alloy in which the more ferromagnetic particles are separated by a less ferromagnetic matrix, the particles may be small enough to constitute a single domain in which boundary formation cannot occur. (The necessary conditions are considered in § 6.) The rotation process alone can then take place, and, as the field is increased, discontinuous as well as continuous changes may occur in the resolved magnetization in the field direction. The course of the changes, that is the form of the I_H, H curve for the particle, depends on its anisotropy. If the particle is spherical, the determining factors will be the magneto-crystalline and strain anisotropy. It may easily be shown that the maximum value of the critical field for a discontinuous change would then be of the order K'/I_0 or $\lambda\sigma_i/I_0$, the first giving values of about 100 (putting $K' = 10^5$, $I_0 = 10^3$), and the second, even with improbably large tensional stress (say $\sigma = 200\,\text{kg.mm.}^{-2}$) about 500. More precise estimates are given in § 7.

The effect of shape anisotropy may be very much greater. Apart from any magneto-crystalline or strain anisotropy, the energy associated with the uniform magnetization of the particle to the quasi-saturation value will depend on the direction of the magnetization vector owing to the directional dependence of the demagnetizing field. For a particle in the form of a prolate spheroid, for example, the energy is greater for magnetization along an equatorial axis than along the polar axis by an amount $\frac{1}{2}(N_b - N_a) I_0^2$ per unit volume, where N_b and N_a are the demagnetization coefficients along the two directions. As will be shown in detail below (§§ 2, 4), in a gradually changing applied field discontinuous changes in the direction of the magnetization may occur at critical values which depend on the orientation of the ellipsoidal particle in the field. The critical field (which corresponds roughly to a value of H_c) may have a value as high as $(N_b - N_a) I_0$. The limiting value, for elongated particles ($N_a \to 0$), is $2\pi I_0$, corresponding, for particles of iron ($I_0 \fallingdotseq 1700$) dispersed in a non-ferromagnetic matrix, to a coercivity value of many thousands. This case is an extreme one, but even for a nearly spherical particle with a dimensional ratio as low as 1·1 the coefficient in place of 2π is approximately 0·47, which still corresponds to a coercivity of several hundreds. In view of the interesting possibilities suggested by such rough estimates of orders of magnitude, it seemed desirable to carry out detailed calculations of the dependence on field of the resolved magnetization of single domain ellipsoidal particles, and also on the behaviour of an assembly of such particles with random orientation. The results of these calculations are presented in this paper. The particles are idealized as ellipsoidal in order to make the mathematical treatment tractable. There is, in fact, no other form offering a practical possibility of combining generality with numerical precision. Uniform magnetization is in general possible only for bodies bounded by surfaces of the second degree. Only in that case can a demagnetization factor be uniquely defined, and the demagnetization relations be given in a relatively simple form which is independent of the magnetic characteristics of the material. Any apparent artificiality is mitigated by the fact that the general ellipsoidal form covers, as an approximation, almost the whole variety of possible shapes for the physical particles, or segregates, which are likely to be of physical interest.

Detailed calculations have been made for particles in the form of spheroids, prolate and oblate. The range of particle shapes covered in this way is from a plate or disk at one extreme to a thin rod or needle at the other. No useful purpose would be served by carrying out the

very heavy calculations which would be required in dealing with particles of general ellipsoidal form, with three unequal axes; the general character of the results which would be obtained is sufficiently clear from those for ellipsoids of revolution. The oblate spheroid is a special limiting case for which, although discontinuities in magnetization may occur, there is no hysteresis. Most of the numerical results can be derived fairly simply from those for the prolate spheroid. The general treatment is given in § 2, the computational procedure is outlined in § 3, and the numerical results for the prolate spheroid are presented in § 4. The oblate spheroid and the general ellipsoid are briefly dealt with in § 5. The conditions necessary for a particle to consist of a single domain are considered in § 6. In § 7, it is shown that, with a suitable modification in the physical interpretation of the non-dimensional parameters used, the numerical results obtained for shape anisotropy of single domain particles are directly applicable to strain and magneto-crystalline anisotropy; and the possible physical bearing of the results as a whole is discussed with reference to various types of ferromagnetic materials.

The general line of inquiry was first embarked upon in 1939, and some preliminary calculations were then made. The work had to be almost entirely abandoned during the war period, and little more than vague references to it have been published (Stoner 1940, 1944, 1945). Systematic work on the problem was resumed at the beginning of 1946.

2. The field dependence of the direction of magnetization of a uniformly magnetized ellipsoid

General

The formal problem to be treated is that of determining the equilibrium direction or directions of magnetization of an ellipsoid whose magnetization is uniform and of constant absolute magnitude as dependent on the magnitude of an applied field and on its direction relative to the principal axes of the ellipsoid. Since there may be more than one direction of magnetization for which, for certain field ranges, the energy is a minimum, it is also necessary to consider the course of the change as the field is changed continuously from a value for which the magnetization direction is uniquely determined. Physically, the assumption of a constant intensity of magnetization, I_0, means that the case treated is that in which the magnitude, as distinct from the direction, of the magnetization is determined essentially by interchange interaction effects, compared with which the effect of an applied field is negligible; and the assumption of uniformity means that the case treated is that in which the shape and size of the ellipsoid are such that domain boundary formation is precluded.

For the general ellipsoid the energy per unit volume associated with the demagnetizing field is

$$E_D = \tfrac{1}{2}I_0^2(N_1\alpha_1'^2+N_2\alpha_2'^2+N_3\alpha_3'^2), \tag{2·1}$$

where α_1', α_2', α_3' are the direction cosines of I_0 with respect to the principal axes, denoted by 1, 2, 3, of the ellipsoid, and N_1, N_2, N_3 are the demagnetization coefficients along these axes. It may be noted that

$$N_1+N_2+N_3 = 4\pi. \tag{2·2}$$

The energy associated with the applied field is

$$E_H = -HI_0\cos\phi, \tag{2·3}$$

where ϕ is the angle between H and I_0. The total energy, apart from constant terms under the conditions assumed, is the sum of E_D and E_H. For a specified field in a given direction with respect to the ellipsoid axes, the energy may be expressed as a function of two variables, and equations for the directions of magnetization corresponding to energy minima may be obtained by standard methods, as outlined in § 5. The labour entailed in obtaining numerical solutions of these equations to cover the H range (in magnitude and direction) effectively would, however, be out of all proportion to the physical interest or value of the results. The physical essentials become clear from a treatment of the two special forms of the general ellipsoid, the prolate and the oblate spheroid. These two forms are very dissimilar in their magnetic behaviour, but the numerical results for one form can be derived without difficulty from those for the other, so it is sufficient to consider in detail only the prolate spheroid.

For the prolate spheroid (polar axis, a, equatorial axis, b) it may be shown (see § 5) that the equilibrium (minimum energy) directions of magnetization lie in the plane defined by the directions of the field and of the polar axis of the ellipsoid. In this plane, let θ be the angle between the positive direction of H and the polar axis, ϕ the angle H, I_0, and ψ the angle between the polar axis and I_0 (see figure 1), so that

$$\phi = \theta + \psi. \tag{2·4}$$

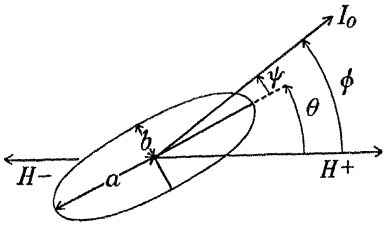

FIGURE 1. Symbols.

The equation (2·1) then reduces to

$$E_D = \tfrac{1}{2} I_0^2 (N_a \cos^2 \psi + N_b \sin^2 \psi), \tag{2·5}$$

and (3·2) gives

$$N_b = 2\pi - \tfrac{1}{2} N_a. \tag{2·6}$$

For the prolate spheroid,

$$N_b > N_a, \quad 2\pi \geqslant N_b > 4\pi/3, \quad 4\pi/3 > N_a \geqslant 0. \tag{2·7}$$

For the sphere and oblate spheroid $N_b = N_a$ and $N_b < N_a$ respectively. The energy associated with the applied field is given by (2·3). By a slight transformation the total (relevant) energy, E', may be expressed as

$$E' = E_D + E_H = \tfrac{1}{4}(N_b + N_a) I_0^2 - \tfrac{1}{4}(N_b - N_a) I_0^2 \cos 2\psi - H I_0 \cos \phi.$$

It is convenient to express the parameters in non-dimensional form by dividing throughout by the positive quantity $(N_b - N_a) I_0^2$, giving for the reduced energy, η',

$$\eta' = \frac{E'}{(N_b - N_a) I_0^2} = \frac{1}{4} \frac{N_b + N_a}{N_b - N_a} - \tfrac{1}{4} \cos 2\psi - \frac{H}{(N_b - N_a) I_0} \cos \phi,$$

or, for the variable part of the energy, η,

$$\eta = -\tfrac{1}{4} \cos 2\psi - h \cos \phi, \tag{2·8}$$

where

$$h = H/(N_b - N_a) I_0.$$

The expression (2·8) for η suggests immediately the physical origin of the two terms contributing to the energy, but, as the consequential relations of greatest physical interest are those between ϕ (or $\cos\phi$), θ (or $\cos\theta$), and h, it is perhaps better to rewrite the equation as

$$\eta = -\tfrac{1}{4}\cos 2(\phi-\theta) - h\cos\phi, \tag{2·9}$$

giving η as $\eta(h,\phi,\theta)$. Treating h and θ as fixed, the stationary values of the function are given by

$$\partial\eta/\partial\phi = \tfrac{1}{2}\sin 2(\phi-\theta) + h\sin\phi = 0, \tag{2·10}$$

and these correspond, respectively, to minima, points of inflexion, or maxima for

$$\partial^2\eta/\partial\phi^2 = \cos 2(\phi-\theta) + h\cos\phi \gtreqless 0. \tag{2·11}$$

The equation (2·10), the central equation of which numerical solutions are required, provides a relation between h and the angles θ and ϕ (or, alternatively, appropriate trigonometrical functions of these angles, such as $\cos\theta$ and $\cos\phi$), and can be regarded as an equation for any one of these three in terms of the other two. The ultimate requirement is a solution giving ϕ (or $\cos\phi$) as $\phi(h,\theta)$, and this is best approached indirectly. As an equation in $\cos\phi$ (or in $\cos\psi$), (2·10) is a quartic, and although the relations between the roots of this equation may be used advantageously in dealing with critical values (see below), the general solution is not only very troublesome to derive, but also, when obtained, not adapted for numerical evaluation. The general procedure adopted, therefore, has been to evaluate h directly from (2·10) for values of ϕ at suitable intervals, for each of a set of suitably spaced values of θ. From the numerical results, values of ϕ for particular values of h may be found by inverse interpolation (see § 3).

The form of the solution of (2·10) for h as $h(\phi)$ is shown for $\theta = 10°$ in figure 2, in which the curves may be referred to as $(h,\phi)_\theta$ contours. The inset shows the form of the corresponding magnetization curve. The term 'magnetization curve' will be used of the curve giving the relation between the resolved value of I_0 in the field direction (or of I_H/I_0, that is $\cos\phi$) and H (or h). For a single ellipsoid there will in general be a component of the magnetization normal to the direction of the field, but it is hardly necessary to consider this component in detail, as it can readily be obtained, if required, from the results for the parallel component. In the case most likely to be of physical interest, that of an assembly of ellipsoids with random orientation, the resultant normal component will be zero.

The physical character of the change corresponding to a discontinuity in the magnetization curve (inset in figure 2) is perhaps most readily appreciated from a set of curves showing the dependence of energy on direction of magnetization for a series of constant values of h, that is $(\eta,\phi)_{h,\theta}$ curves, of which a number are given in figure 3 for $\theta = 10°$. Beginning with a positive h for which there is a unique energy minimum, as h decreases the equilibrium values of ϕ are defined by a continuous sequence of energy minima until, at a critical value, h_0, of the field, the minimum and approaching maximum coincide, and subsequently disappear. As h approaches h_0 from the positive side, ϕ approaches ϕ_0, and as h passes through h_0, ϕ changes continuously but 'spontaneously' to the value ϕ_0', corresponding to a new energy minimum, with a continuous series of minima on either side, that is for greater and smaller values of h. The process can be readily followed from the figures, and the relation between figure 2 and figure 3 hardly requires detailed commentary.

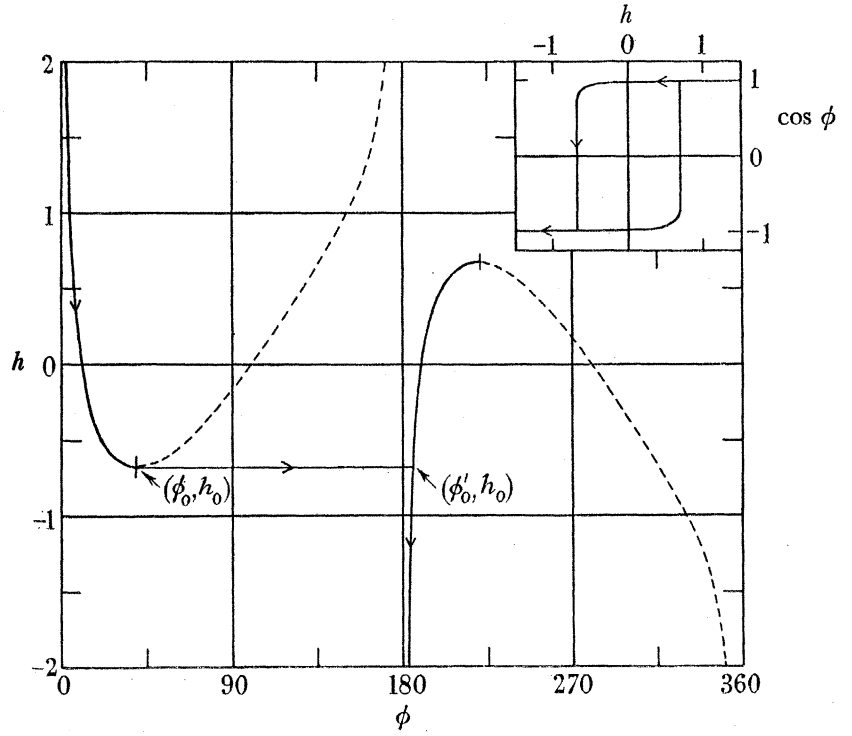

FIGURE 2. Relation between field and direction of magnetization for stationary values of the energy for prolate spheroid oriented at 10° to the field. ϕ, angle between H and I_0; $h = H/(N_b - N_a)\, I_0$; N_a, N_b demagnetizing coefficients along polar and equatorial axes. The full and broken parts of the curves correspond to energy minima and maxima respectively. The arrows indicate the course of the change in ϕ as h is reduced from a positive value greater than $|h_0|$. ($|h_0| = 0{\cdot}6738$, $\phi_0 = 39{\cdot}28°$, $\phi' = 180° + 5{\cdot}97°$.) The corresponding magnetization curve is shown in the inset diagram.

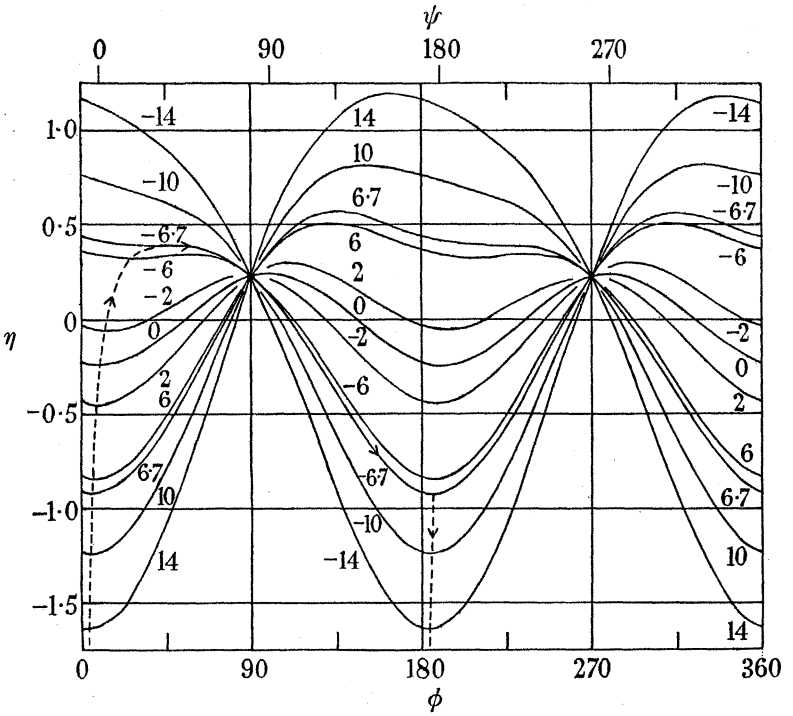

FIGURE 3. Relation between energy and direction of magnetization for constant field for a prolate spheroid oriented at 10° to the field. The values of $10h$ (h, reduced field) are given on the curves. ϕ, angle between H and I_0. η, reduced energy (see equation 2·8). The arrows indicate the course of the change in ϕ as h is reduced from a positive value greater than $|h_0|$. See also figure 2.

Critical values

For a given value of θ, the critical values of h and ϕ, say h_0 and ϕ_0, for which the change in the equilibrium direction of magnetization becomes discontinuous as the field changes continuously, are such that, the first partial derivative of η with respect to ϕ being zero, the second passes through zero from a positive to a negative value. The three quantities h_0, ϕ_0 and θ are then related by the simultaneous equations (2·10) and (2·11) (with the equality sign); eliminating any one, a relation can be obtained between the other two. It is simplest to eliminate h, giving

$$\tan 2(\phi_0-\theta) = 2\tan\phi_0,$$

or, in terms of ψ_0,

$$\tan 2\psi_0 = 2\tan(\psi_0+\theta),$$

giving

$$\tan^3\psi_0 = \tan\theta. \tag{2·12}$$

It is convenient to write

$$\tan^{\frac{1}{3}}\theta = \tan\psi_0 = t. \tag{2·13}$$

The value of ϕ_0 is then given by

$$\phi_0 = \psi_0+\theta = \theta+\tan^{-1}t, \tag{2·14}$$

and

$$\tan\phi_0 = t/(1-t^2). \tag{2·15}$$

With the three quantities h_0, ϕ_0 and θ (or trigonometrical ratios for the angles) there are essentially six relations giving any one explicitly in terms of a second. These relations can be obtained by straightforward manipulation of the equations. The relation (2·15), in view of (2·13), may be symbolized as the $\phi_0(\theta)$ relation, the inverse relation being

$$t = \frac{1}{2\tan\phi_0}\{\pm(1+4\tan^2\phi_0)^{\frac{1}{2}}-1\}. \tag{2·15a}$$

The equations relating h_0 and θ are

$$h_0 = -(1-t^2+t^4)^{\frac{1}{2}}/(1+t^2), \tag{2·16}$$

and

$$t = \frac{1}{2(1-h_0^2)^{\frac{1}{2}}}\{3^{\frac{1}{2}}\mp(4h_0^2-1)^{\frac{1}{2}}\}, \tag{2·16a}$$

and those relating h_0 and ϕ_0,

$$h_0 = -\{(1+\tan^2\phi_0)/(1+4\tan^2\phi_0)\}^{\frac{1}{2}}, \tag{2·17}$$

and

$$\tan\phi_0 = \{(1-h_0^2)/(4h_0^2-1)\}^{\frac{1}{2}}. \tag{2·17a}$$

When h decreases from a positive value greater than $+|h_0|$, h_0 is negative, and when h increases from a negative value less than $-|h_0|$, h_0 is positive. Equation (2·16a) or (2·17a) shows at once that $|h_0|$ falls in the range $\frac{1}{2}\leqslant|h_0|\leqslant 1$, the value $|h_0|=1$ occurring, from (2·16a), for $\theta = 0°$ or 90°, and $|h_0| = \frac{1}{2}$ for $\theta = 45°$. From (2·16), noting that the expression is unchanged on substituting $1/t$ for t, it is apparent that h_0 is the same for pairs of values of θ symmetrical about $\theta = 45°$. It is not necessary to deal formally with the question of the physical significance of multiple roots, or with the method of selection of the relevant root, as this, in practice, is quite clear from the general sequence of values. Numerical results are given in § 4.

At a critical value, h_0, of the field, for a given value of θ, the angle of magnetization changes from ϕ_0 to ϕ_0' (or from ψ_0 to ψ_0'). The relation most readily obtained is that between $\tan\psi_0'$

and t. Expressing (2·10) in terms of $\tan\psi$ and t, and substituting for h from (2·16), a quartic in $\tan\psi$ is obtained. This quartic has two coincident roots, given by (2·13), and can be factorized. The roots of the quadratic so obtained are found to be

$$\tan\psi_0' = t\{\pm(1+t^2+t^4)^{\frac{1}{2}} - (1+t^2)\}, \tag{2·18}$$

one root corresponding to an energy minimum, the other to an energy maximum. (The relations will be clear from figure 2.) The relevant root (energy minimum) may be found by inspection. Equations analogous to (2·15) and (2·17) are too cumbersome to be useful, and in general it is simpler to work via (2·18) rather than to obtain direct relations involving ϕ_0. If, for example, ϕ_0' is required for a given value, h_0, of h, the value of t is first obtained from (2·16a), and substituted in (2·18); from $\tan\psi_0'$, ψ_0' is found, and hence ϕ_0' from (2·4). It may be noted that the physical character of the change is shown by the values of ϕ_0 and ϕ_0' (or of ψ_0 and ψ_0'), and the magnitude of the discontinuity in the (resolved) magnetization by $\cos\phi_0' - \cos\phi_0$.

Random orientation. The $(\phi,\theta)_h$ diagram

The mean resolved value in the positive field direction of the magnetization of an assembly of similar spheroids with polar axes orientated at random in a particular field h is given by

$$\overline{\cos\phi} = I_H/I_0 = \int_0^{\pi/2} 2\pi\cos\phi\,\sin\theta\,d\theta \Big/ \int_0^{\pi/2} 2\pi\sin\theta\,d\theta,$$
$$= \int_0^{\pi/2} \cos\phi\,\sin\theta\,d\theta, \tag{2·19}$$

where the value of ϕ to be taken, when there are alternative values, is that consistent with the previous history. The case explicitly considered here is that for descending h, h decreasing from a positive value greater than $+1$. (The ascending magnetization curve is immediately obtainable by symmetry. Subsidiary hysteresis loops could be obtained, if required, by a suitable adaptation of the general method.) The value of the integral (2·19) is required for a series of values of h, a suitable set being from $+1{\cdot}5$ to $-1{\cdot}0$ at intervals of $0{\cdot}1$. Except in special cases ($h = 0, \pm\frac{1}{2}$), the integral must be evaluated numerically, and if the more convenient methods of numerical integration are to be used, the behaviour of the integrand beyond the limits of integration must be examined. For this reason, and also because of the intrinsic interest of the relations shown, it is appropriate to consider the general character of the dependence of ϕ on θ at constant h.

For stationary values of the energy, the values of ϕ, θ and h are related by (2·10). The function represented has a period of 2π in ϕ, and of π in θ. Constant h contours for $h = 0$, $\pm 0{\cdot}5$, $\pm 0{\cdot}6$ and $\pm 1{\cdot}0$ are shown in figure 4 for the range $0 \leqslant \theta \leqslant 180°$, $0 \leqslant \phi \leqslant 360°$, the portions of contours corresponding to energy minima and maxima being shown by full and broken curves respectively.

In explanation of figure 4, it will be sufficient to consider the contours for $h = -0{\cdot}6$ (labelled -6) for the range $0 \leqslant \theta \leqslant 90°$. If h has decreased to $-0{\cdot}6$ from a positive value greater than 1, the dependence of ϕ on θ is shown by the full curve in area (1) for $0 \leqslant \theta < 16{\cdot}6°$, that in area (3) for $16{\cdot}6 < \theta < 73{\cdot}4°$, and that in area (2) for $73{\cdot}4 < \theta \leqslant 90°$. (If, on the other hand, h had increased to $-0{\cdot}6$ from a value less than -1, the dependence of ϕ on θ for the whole range $0 \leqslant \theta \leqslant 90°$ would be that shown by the full curve in area (3).) It should perhaps

be noted that the sequence of values of ϕ for a stationary set of ellipsoids distributed in angle over the θ range from 0 to 90° at a value of h reached by reduction from a positive value is not necessarily the same as the sequence of values which would be obtained if a single ellipsoid were rotated through this angular range with h remaining constant. As an example, in the rotational case, a representative point being on the $h = -0{\cdot}6$ contour in area (3) would, with h remaining at $-0{\cdot}6$, move along this contour as θ increased, leaving it at the critical value in area (7) ($\theta = 90 + 16{\cdot}6°$). For the case under consideration, the representative point moves along the contour in area (3) only over the θ range from 16·6° to 73·4°, as already stated.

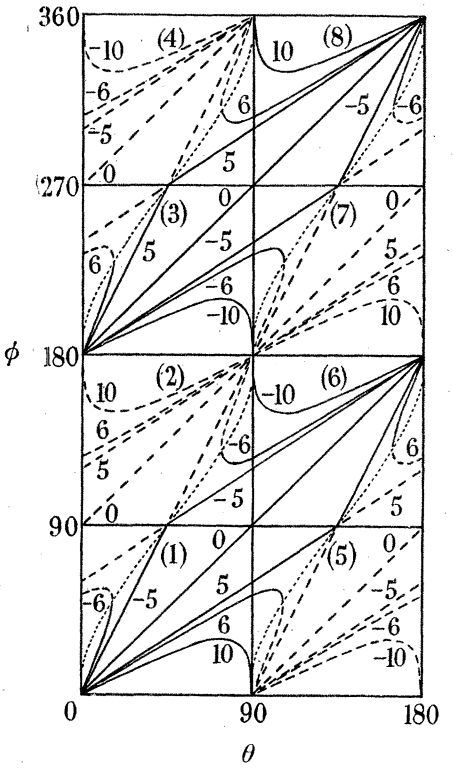

FIGURE 4. Dependence of direction of magnetization, ϕ, on orientation, θ, of polar axis of ellipsoid with respect to the field. Full curves, energy minima. Broken curves, energy maxima. The numbers on the curves give the values of $10h$. The dotted curves give the critical angles ϕ_0. The numbers in brackets, (1) to (8), are for reference to the corresponding areas in the diagram representing 90° ranges in θ and ϕ.

The symmetry relations which are apparent in figure 4 may be expressed in a number of ways. The most useful set of independent relations is indicated in table 1. These relations may be verified by substitution in (2·10) and (2·11).

With the aid of these relations, the $(\phi, \theta)_h$ curves for the entire angular range (180° in θ, 360° in ϕ) may be derived at once from those for any angular range of 90° in θ and 90° in ϕ, i.e. those in any one of the eight numbered areas in figure 4. Assuming the associated values of ϕ and θ to have been found corresponding to both energy minima and maxima for the required series of positive and negative values of h for $0 \leqslant \theta \leqslant 90°$, $0 \leqslant \phi \leqslant 90°$ (area (1) in figure 4), the contours in area (2) may be drawn using the relation (*b*) in table 1. Those in areas (3) and (4) are then obtainable using (*c*), and finally, using (*d*) the contours in the

areas (5) to (8). (As an alternative, the whole contour diagram may be built up from the energy minima contours in the range $0 \leqslant \theta \leqslant 90°$, $0 \leqslant \phi \leqslant 180°$, that is the contours in the right-hand lower half of areas (1) and (2), bounded by the curve giving the critical angle, ϕ_0.) A more complete contour diagram, corresponding to area (1) in figure 4, is shown in figure 5.

TABLE 1. RELATIONS BETWEEN ANGLES CORRESPONDING TO ENERGY MINIMA AND MAXIMA

θ, angle between polar axis of ellipsoid and positive direction of field.
ϕ, angle between magnetization vector, I_0, and positive direction of field.

			$+\lvert h_1 \rvert$		$-\lvert h_1 \rvert$	
	θ		min.	max.	min.	max.
(a)	θ_1		ϕ_+	ϕ'_+	ϕ_-	ϕ'_-
(b)	$\frac{1}{2}\pi - \theta_1$	$\pi-$	ϕ'_+	ϕ_+	ϕ'_-	ϕ_-
(c)	θ_1	$\pi+$	ϕ_-	ϕ'_-	ϕ_+	ϕ'_+
(d)	$\frac{1}{2}\pi + \theta_1$		ϕ'_-	ϕ_-	ϕ'_+	ϕ_+

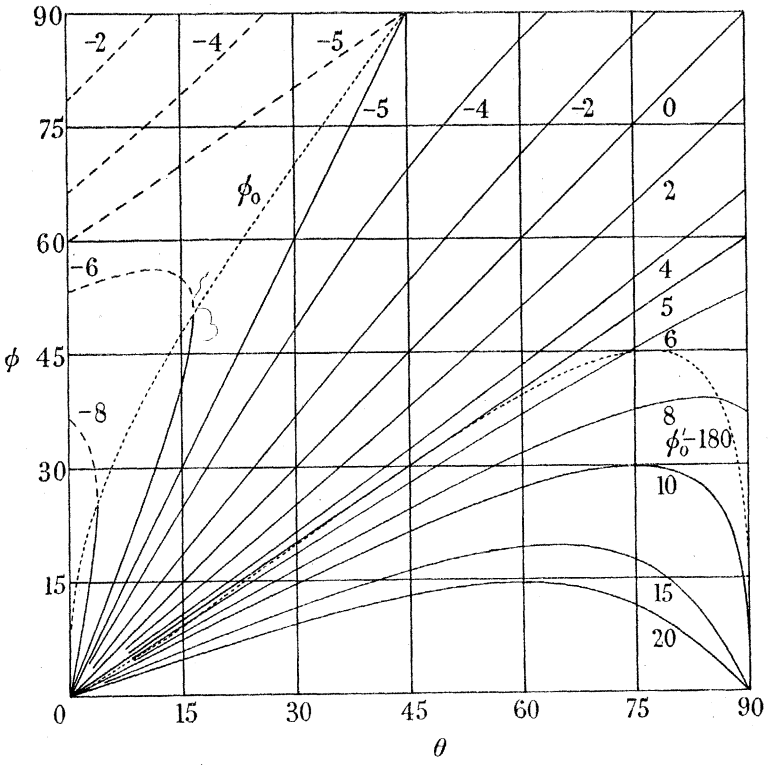

FIGURE 5. Dependence of direction of magnetization, ϕ, on orientation, θ, of polar axis of ellipsoid with respect to the field, for $0 \leqslant \theta \leqslant 90°$, $0 \leqslant \phi \leqslant 90°$. Full curves, energy minima. Broken curves, energy maxima. The numbers on the curves give the values of $10h$. The dotted curves give ϕ_0 and $\phi'_0 - 180$. See also figure 4 and table 1.

As would be anticipated from the character of the contours in figures 4 and 5, the numerical evaluation of the integral (2·19), using the numerical tables from which the figures are drawn, presents no difficulty for positive values of h not too near to unity, or for negative values in the range $-\frac{1}{2} \leqslant h < 0$. For the range $-1 < h < -\frac{1}{2}$, the integration is very troublesome, not only owing to the integral being made up of three parts with 'awkward' limits,

but also owing to the behaviour of the integrand near these limits. Similar behaviour is encountered in connexion with the integration for $h = 1$, associated with the fact that for $\theta \to 90°$, $d\phi/d\theta \to \infty$. The method of dealing with these integrations is outlined in § 3, and the results, along with others, are given in § 4.

Solutions of equation for stationary energy in special cases

Useful explicit solutions of the stationary energy equation (2·10), treating ϕ as the unknown, are obtainable in the forms $\phi(h)$ and $\phi(\theta)$ for only a limited number of particular values of θ and h respectively. The results obtained for $\theta = 0$, 45 and 90°, and for $h = 0$ and $\pm\frac{1}{2}$ are summarized below. The solutions correspond to energy minima, denoted by m, or energy maxima, denoted by M, as determinable from (2·11). The character of the solution over the range of the appropriate variable, or of ϕ, is indicated by inserting (m) or (M) between the limiting values, the total range so characterized and the particular solutions chosen for the angles, being such as to cover the range of figure 4 ($0 \leqslant \theta \leqslant 180°$, $0 \leqslant \phi \leqslant 360°$) in the most convenient way. The angles are given in degrees, as being more convenient practically, though the angles actually specified are all multiples of simple fractional parts of π. The significance of the solutions will be readily apparent from inspection of figure 2 or figure 4.

$\theta = 0$: (1) $\sin\phi = 0$

$\phi = 0$ $\quad h \; +\infty \; (m) \; -1 \; (M) \; -\infty,$

$\phi = 180$ $\quad h \; -\infty \; (m) \; +1 \; (M) \; +\infty;$

(2) $\cos\phi = -h$ $\quad \begin{cases} h & +1 \; (M) & -1 \; (M) & +1, \\ \phi & 0 & 180 & 360. \end{cases}$

$\theta = 45$: $h = \frac{1}{2}\operatorname{cosec}\phi - \sin\phi$

(1) $\sin\phi = \frac{1}{2}\{(h^2+2)^{\frac{1}{2}} - h\}$ $\quad \begin{cases} h & +\infty \; (m) & -\frac{1}{2} \; (M) & +\infty, \\ \phi & 0 & 90 & 180; \end{cases}$

(2) $\sin\phi = -\frac{1}{2}\{(h^2+2)^{\frac{1}{2}} + h\}$ $\quad \begin{cases} h & -\infty \; (m) & +\frac{1}{2} \; (M) & -\infty, \\ \phi & 180 & 270 & 360. \end{cases}$

$\theta = 90$: (1) $\sin\phi = 0$

$\phi = 0$ $\quad h \; +\infty \; (m) \; +1 \; (M) \; -\infty,$

$\phi = 180$ $\quad h \; -\infty \; (m) \; -1 \; (M) \; +\infty;$

(2) $\cos\phi = h$ $\quad \begin{cases} h & +1 \; (m) & -1 \; (m) & +1, \\ \phi & 0 & 180 & 360. \end{cases}$

$h = 0$: $\sin 2(\phi - \theta) = 0,$

$\phi = \theta(m), \quad \phi = \theta + 90(M), \quad \phi = \theta + 180(m), \quad \phi = \theta + 270(M).$

$h = +\frac{1}{2}$: (1) $\sin(\frac{3}{2}\phi - \theta) = 0$

$\phi = \frac{2}{3}\theta \quad \theta \quad 0(m) \; 135(M) \; 270,$

$\phi = \frac{2}{3}\theta + 120 \quad \theta \quad -45(M) \; 90(M) \; 225,$

$\phi = \frac{2}{3}\theta + 240 \quad \theta \quad -90(M) \; 45(m) \; 180;$

(2) $\cos(\frac{1}{2}\phi - \theta) = 0$

$\phi = 2\theta + 180 \quad \theta \quad 0(m) \; 45(M) \; 90,$

$\phi = 2\theta - 180 \quad \theta \quad 90(M) \; 135(m) \; 180.$

$h=-\frac{1}{2}$: (1) $\cos(\frac{3}{2}\phi-\theta)=0$

$\phi=\frac{2}{3}\theta-\ 60$ $\quad\theta\quad -45(M)\ \ 90(M)\ 225,$

$\phi=\frac{2}{3}\theta+\ 60$ $\quad\theta\quad -90(M)\ \ 45(m)\ \ 180,$

$\phi=\frac{2}{3}\theta+180$ $\quad\theta\quad\ \ 0(m)\ \ 135(M)\ 270,$

$\phi=\frac{2}{3}\theta+300$ $\quad\theta\quad -45(M)\ \ 90(M)\ 225;$

(2) $\sin(\frac{1}{2}\phi-\theta)=0$

$\phi=2\theta$ $\quad\theta\quad\ \ 0(m)\ \ 45(M)\ \ 90,$

$90(M)\ 135(m)\ \ 180.$

An intercomparison of the solutions and their character for $\theta=0$ and 90°, and for $h=+\frac{1}{2}$ and $-\frac{1}{2}$, exemplifies very clearly the relations given in table 1.

Using the explicit solutions given above for $h=0,\ \pm\frac{1}{2}$, the integration in (2·19) may be carried out and the mean resolved magnetization for ellipsoids orientated at random obtained. The following results apply for h decreasing from a positive value greater than 1.

$$h=0: \quad \overline{\cos\phi}=\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta=+\tfrac{1}{2}.$$

$$h=+\tfrac{1}{2}: \quad \overline{\cos\phi}=\int_0^{\pi/2}\cos\tfrac{2}{3}\theta\sin\theta\,d\theta.$$

The integration may be effected by using $\cos\frac{1}{3}\theta$ as the variable, or otherwise, and gives

$$\overline{\cos\phi}=\tfrac{3}{5}(3-3^{\frac{1}{2}})=0{\cdot}760\,770.$$

$$h=-\tfrac{1}{2}: \quad \overline{\cos\phi}=\int_0^{\pi/4}\cos 2\theta\sin\theta\,d\theta+\int_{\pi/4}^{\pi/2}\cos(\tfrac{2}{3}\theta+\pi/3)\sin\theta\,d\theta,$$

$$=\tfrac{1}{3}(2^{\frac{1}{2}}-1)-\tfrac{3}{5}(3^{\frac{1}{2}}-2^{\frac{1}{2}}),$$

$$=0{\cdot}138\,071-0{\cdot}190\,702=-0{\cdot}052\,631.$$

For h increasing from a negative value less than -1, the values of $\cos\phi$ for $h=-\frac{1}{2}$ and $h=+\frac{1}{2}$ are the values given above for $h=+\frac{1}{2}$ and $h=-\frac{1}{2}$ with reversed signs, i.e. $-0{\cdot}760\,770$ and $+0{\cdot}052\,631$ respectively.

3. Computational details

The central aim in the computational work has been to draw up a set of tables giving the resolved magnetization as a function of the field for a suitable series of values of the orientation of the polar axis of the ellipsoid relative to the field; that is giving $\cos\phi$ as a function of h for a series of values of θ (see table 3, § 4). Values of $\cos\phi$ are given to 4 places of decimals. This accuracy may seem excessive for the purposes of such indirect comparison with experiment as may be possible, but a lower accuracy would have left uncertainties as to some of the interesting detailed characteristics of the behaviour. The actual calculations were made to at least 5 places of decimals, and rounded values are given in the tables. Except in the tables of critical values (tables 4 and 5), values of angles are given to 0·01°, and values of h to 4 places of decimals. For the trigonometrical functions, the convenient tables of Lohse (ed. Neugebauer 1935) (5 place, interval 0·01°) were largely used, supplemented by those of Peters (1942) (7 place, interval 0·001°). These latter tables were particularly useful in

the calculation of critical values. Barlow's tables (ed. Comrie 1941) were used for powers and roots. Most of the numerical work was carried out with the aid of a Brunsviga calculating machine.

Inverse interpolation

As extensive inverse interpolation was required in the work it is perhaps desirable to indicate the method followed. As already stated (§ 2), the basic equation (2·10), namely

$$\tfrac{1}{2}\sin 2(\phi-\theta)+h\sin\phi=0, \tag{3·1}$$

was first solved for h, values of h being found for $\theta = 0(10)\,90$, and ϕ at intervals of 1° over the requisite range. Owing to the character of the variation of h with ϕ (see, for example, figure 2) the tables so obtained do not in general lend themselves to inverse interpolation to give ϕ as a function of h to the accuracy required in a single process (say by the two-machine method of Comrie) without a much smaller interval than that chosen. As the number of inverse interpolates required for each value of θ is not usually large, however, and small interval $h(\phi)_\theta$ tables would be of little direct value, it was considered more economical to use the tables merely for the purpose of obtaining rough estimates of the inverse interpolates, to form the starting point for more accurate determinations by an adaptation of the usual method of trial. An estimate of ϕ having been obtained for a given value of h, either by linear interpolation or allowing for second differences, two values ϕ_1 and ϕ_2 are taken which bracket the estimated value at as small an interval as is consistent with ensuring that the correct value of ϕ for the given h is also bracketed. A usual procedure is then to calculate the corresponding values of h, say h_1 and h_2, from (3·1), and to obtain the value of ϕ for the given h by linear interpolation; the process being repeated, if necessary. A modification adopted was as follows. Writing $f=\tfrac{1}{2}\sin 2(\phi-\theta)$, and $F=-h\sin\phi$, where h is the value for which ϕ is required, the values of f_1, f_2, F_1 and F_2 were calculated, and with $f_1=\tfrac{1}{2}\sin 2(\phi_1-\theta)$, $F_1=-h\sin\phi_1$, etc., the following results are obtained, where differences are indicated by $\omega, \alpha, \beta, \gamma, \delta$,

$$\begin{array}{lllll} \phi_1 & & f_1 & (\gamma) & F_1, \\ (\omega) & & (\alpha) & & (\beta), \\ \phi_2 & & f_2 & (\delta) & F_2. \end{array}$$

Since, for the correct value of ϕ, say $\phi=\phi_1+n\omega, f=F$, n is given by

$$n=\gamma/(\alpha-\beta). \tag{3·2}$$

Although this modification of the usual procedure is formally trivial, practically it effects a considerable saving of time, particularly in view of the fact that the values of h are usually 1 or 2 figure numbers, and that the difference between ϕ_1 and ϕ_2 can usually be made, from the first estimate, sufficiently small for the calculation of (3·2) to be made mentally.

Numerical integration

For the calculation of the mean magnetization at a given value of h of an assembly of ellipsoids orientated at random in the field (see table 6, § 4), the value of the integral in (2·19) is required, namely

$$\overline{\cos\phi}=\int_0^{\pi/2}\cos\phi\,\sin\theta\,d\theta. \tag{3·3}$$

From table 2, giving $\cos\phi$ as a function of h at a series of values of θ, tables may be drawn up giving $(\cos\phi)_h$ for $\theta = 0$ (10°) 90°, and a series of equally spaced values of the integrand obtained by multiplying by $\sin\theta$. Where the integrand is a continuous function of θ over the range of integration, that is for $|h| < \frac{1}{2}$, for $\frac{1}{2} < h \leqslant 1$, and for $|h| \geqslant 1$, the integration can in principle be effected numerically, using the Newton-Gregory formula, or, if the continuation of the function outside the θ range from 0 to 90° is known and is of a suitable character, the more convenient central difference formula. For $|h| < \frac{1}{2}$, the integration can in fact be effected with the required accuracy with the intervals in θ as large as 10°, but smaller intervals (5 or 2·5°) are required for $\frac{1}{2} < h < 1$, and also for $|h|$ greater than, and close to 1. (The difficulties arise from the behaviour of the differences of the integrand as θ approaches 90°, the general character of which can be appreciated, without detailed description, from the forms of the curves in figures 4 and 5.) For $|h| = 1$, accurate integration by this method is virtually impracticable, since for $\theta \to 90°$, $(d\phi/d\theta) \to -\infty$. Similar difficulties arise for all values of h in the range $-1 < h < -\frac{1}{2}$ at the critical values of θ. The integrals extend over three continuous ranges, as may be exemplified for $h = -0{\cdot}6$ (see figures 4 and 5).

		(1)	(2)	(3)
$h = -0{\cdot}6$:	θ	0—16·59	16·59— 73·41	73·41— 90
	ϕ	0—50·34	190·36—224·56	129·66—126·87

The contribution from range (2) is readily evaluated; it is simply the negative of the corresponding integral for $h = +0{\cdot}6$ between the particular limits. Difficulties arise at the upper limit of range (1), and the lower limit of range (3). These can be overcome by using a suitable transformation of the integral.

The contribution to the required integral over any continuous range between θ_1 and θ_2 may be expressed in the following three forms:

$$\Delta\overline{\cos\phi} = \int_{\theta_1}^{\theta_2} \cos\phi \sin\theta \, d\theta, \tag{3·4}$$

$$= -\int_{\cos\theta_1}^{\cos\theta_2} \cos\phi \, d(\cos\theta), \tag{3·5}$$

$$= -\Big|\cos\phi \cos\theta\Big|_{\theta_1}^{\theta_2} + \int_{\cos\phi_1}^{\cos\phi_2} \cos\theta \, d(\cos\phi). \tag{3·6}$$

Over a portion of a range in which $\cos\phi$ passes through (or to) a maximum or minimum, or, more generally, varies gradually, the form (3·5) is suitable; where $\cos\theta$ varies gradually, the form (3·6). (The term 'gradually' is used loosely. The essential criterion in the choice of a suitable form is the behaviour of the differences of the integrand when tabulated at suitable equal intervals in the integration variable.) Thus, for $h = 1$ (see figure 4), the form (3·6) was used with entries 1·00 (0·01) 0·89 for $\cos\phi$ to cover the θ ranges 0·00 to 60·01, and 84·24 to 90·00; and the form (3·5) with entries 0·50 (0·05) 0·10 for $\cos\theta$ to cover the θ range 60·00 to 84·26. The positive or negative contributions from the unavoidable small gaps or overlaps may be readily estimated, in a number of ways, with the requisite precision. For $h = -0{\cdot}6$ (exemplifying the procedure for $-1 < h < -\frac{1}{2}$), the form (3·6) was used for range (2) (and incidentally for the complete integration for $h = +0{\cdot}6$) and for range (1), and for range (3) a combination of the forms (3·5) and (3·6). Interpolation is necessary when the

limits, such as those imposed by the critical values of θ or ϕ, do not coincide with tabular entries.

The standard tables (tables 2, 3) give ϕ or $\cos\phi$ as a function of θ for given h, and in carrying out integrations using the forms (3·5) or (3·6), direct or inverse interpolation is necessary in drawing up the requisite tables of $\cos\phi$ as a function of $\cos\theta$ or of $\cos\theta$ as a function of $\cos\phi$. Since the intervals are small, however, once the first few entries have been obtained by the method indicated above, subsequent estimates may be readily made with considerable accuracy by forward extrapolation using the part of the table already constructed.

It will be apparent that the labour entailed in computing the integrals corresponding to the mean magnetization for ellipsoids orientated at random for the four h values from $-0{\cdot}6$ to $-0{\cdot}9$ (table 6, § 4) is quite formidable, even allowing for the fact that the integrals for the corresponding positive values of h are obtained incidentally. It must be stressed, however, that a procedure similar to that outlined is necessary if uncertainties are to be avoided. Even if an accuracy of 0·001 rather than of 0·0001 in $\cos\phi$ had been aimed at, it is doubtful whether a direct attack on the integrals (using the form (3·4) exclusively) would have been satisfactory. The type of procedure adopted could clearly be applied with advantage to many problems involving inverse interpolation and numerical integration other than the particular one considered here. It may be mentioned that a number of series expansions were obtained to cover the neighbourhood of 'awkward' points in the hope that they could be utilized in carrying out the integrations. For this purpose, however, they turned out to be of no practical value.

4. Numerical results for prolate spheroid

Numerical results for prolate spheroids, obtained by the methods outlined in § 3, are given in this section, with such brief explanation of, and commentary on, the tables and diagrams as seems necessary.

Variation with field of direction of magnetization (table 2)

The table gives the angle, ϕ, between the magnetization vector, I_0, and the positive direction of the field (see figure 1) for descending field, for a number of orientations of a prolate spheroid. The orientations are specified by the angle, θ, between the polar axis and the positive direction of the field. The reduced field, h, is related to the field, H, by

$$h = H/(N_b - N_a)\, I_0 \quad \text{(see (2·8))},$$

where N_a and N_b are the demagnetization coefficients along the polar and equatorial axes respectively. For ferromagnetic, single-domain, ellipsoids, dispersed in a non-ferromagnetic matrix, I_0 is the spontaneous (or quasi-saturation) magnetization per unit volume of the ferromagnetic (not the mean value for the ferromagnetic and the non-ferromagnetic matrix). Values are tabulated for $h = 2{\cdot}0$, and at intervals of 0·1 downwards from $h = 1{\cdot}5$ to the tabular value preceding the value of h at which the discontinuous change occurs in the equilibrium direction of magnetization. The critical value of h, h_0, the critical angle, ϕ_0, and the final angle, ϕ_0', after completion of the discontinuous change, are shown at the bottom of the table. For negative values of h less than h_0 (e.g. for $\theta = 10°$, values less than

$-0{\cdot}6738$), the angles are those listed for the corresponding positive values increased by 180°. For h increasing from an initial negative value less than h_0, and having the values $-2{\cdot}0$, $-1{\cdot}5$, $-1{\cdot}4$, ..., the sequence of angles is that listed for $h = 2{\cdot}0$, $1{\cdot}5$, $1{\cdot}4$, ..., increased by 180°. It is unnecessary to tabulate values for $\theta = 0°$ (the spheroid orientated with the polar axis parallel to the field), since for h decreasing, $\phi = 0$ for all $h > -1$. Further, $\phi_0 = 0$, $\phi_0' - 180 = 0$, and $-h_0 = 1$.

TABLE 2. DEPENDENCE OF ϕ ON h AND θ FOR PROLATE SPHEROID

h \ θ	10	20	30	40	45	50	60	70	80	90
2·0	3·31	6·51	9·46	11·97	12·99	13·79	14·48	13·25	8·76	0·00
1·5	3·98	7·87	11·55	14·86	16·31	17·56	19·25	19·03	14·63	0·00
1·4	4·15	8·21	12·07	15·91	17·16	18·54	20·52	20·68	16·63	0·00
1·3	4·33	8·58	12·65	16·39	18·09	19·61	21·93	22·53	19·06	0·00
1·2	4·53	8·99	13·28	17·27	19·11	20·79	23·49	24·62	21·99	0·00
1·1	4·75	9·43	13·97	18·24	20·23	22·09	25·21	26·97	25·43	0·00
1·0	4·99	9·92	14·72	19·30	21·47	23·52	27·12	29·58	29·36	0·00
0·9	5·26	10·46	15·56	20·48	22·75	25·10	29·23	32·47	33·71	25·84
0·8	5·55	11·06	16·49	21·79	24·36	26·86	31·56	35·63	38·37	36·87
0·7	5·88	11·73	17·53	23·24	26·04	28·80	34·12	39·05	43·23	45·57
0·6	6·25	12·48	18·69	24·85	27·91	30·95	36·93	42·74	48·24	53·13
0·5	6·67	13·33	20·00	26·67	30·00	33·33	40·00	46·67	53·33	60·00
0·4	7·14	14·30	21·48	28·70	32·33	35·98	43·35	50·83	58·50	66·42
0·3	7·47	15·41	23·17	31·00	34·95	38·93	46·99	55·23	63·73	72·54
0·2	8·34	16·70	25·11	33·60	37·89	42·21	50·95	59·88	69·03	78·46
0·1	9·09	18·21	27·36	36·58	41·21	45·87	55·27	64·78	74·45	84·26
0·0	10·00	20·00	30·00	40·00	45·00	50·00	60·00	70·00	80·00	90·00
−0·1	11·10	22·16	33·14	43·99	49·36	54·70	65·23	75·58	85·75	95·74
−0·2	12·48	24·84	36·96	48·75	54·50	60·15	71·12	81·66	91·78	101·54
−0·3	14·25	28·25	42·78	54·65	60·79	66·72	77·97	88·43	98·22	107·46
−0·4	16·61	32·86	48·36	62·64	69·20	75·36	86·49	96·33	105·26	113·58
−0·5	20·00	40·00	60·00	80·00	90·00	93·33	100·00	106·67	113·33	120·00
−0·6	25·64	—	—	—	—	—	—	—	123·65	126·87
−0·7	—	—	—	—	—	—	—	—	—	134·43
−0·8	—	—	—	—	—	—	—	—	—	143·13
−0·9	—	—	—	—	—	—	—	—	—	154·16
−1·0	—	—	—	—	—	—	—	—	—	180·00
ϕ_0	39·28	55·53	69·78	83·33	90·00	96·67	110·22	124·47	140·71	180·00
$\phi_0' - 180$	5·97	12·70	19·67	26·62	30·00	33·27	39·24	43·75	44·53	0·00
$-h_0$	0·6738	0·5736	0·5240	0·5026	0·5000	0·5026	0·5240	0·5736	0·6738	1·0000

Variation with field of resolved magnetization (table 3)

The resolved value of the magnetization in the positive direction of the field is given by $I_0 \cos\phi$. In table 3 values of $\cos\phi$ are tabulated at the same intervals in θ and in h as in table 2. The data in the table, including the values for $\cos\phi_0$ and $\cos\phi_0'$ at the bottom, together with the value of h_0 from table 2, provide a detailed numerical specification of the reduced magnetization curve for the various orientations of a prolate spheroid in a field. For negative fields increasing from a value less than h_0, the sequence of values of $\cos\phi$ for $h = -2{\cdot}0$, $-1{\cdot}5$, $-1{\cdot}4$, ..., is the negative of the sequence shown for $h = 2{\cdot}0$, $1{\cdot}5$, $1{\cdot}4$, ... The relations will

be clear from the examples of magnetization curves shown in figure 6. For $\theta = 0$, $\cos\phi = 1$ for the h values given in the table, $\cos\phi_0 = 1$, $\cos\phi_0' = -1$.

It is to be noted that the reduced critical field, h_0, is not necessarily the same as the reduced coercivity, h_c (corresponding to ${}_IH_c$, the value of H at which the intensity of magnetization is reduced to zero). For $0 \leqslant \theta < 45°$, $\cos\phi$, and so the resolved value of I, passes through zero in the course of the jump, and h_c is equal to h_0. For $45 < \theta < 90°$, the critical value $\cos\phi_0$ is negative, and $|h_c| < |h_0|$. An approximate value of h_c may be obtained by inverse interpolation in tables 2 or 3. For $\theta = 45°$, the jump begins at $\cos\phi = 0$ ($\phi = 90$). For $\theta = 90$, the jump becomes zero, and the magnetization curve is reversible, though with a discontinuity in the rate of change of magnetization with field at both $h = +1$ and $h = -1$.

TABLE 3. DEPENDENCE OF $\cos\phi$ ON h AND θ FOR PROLATE SPHEROID

h \ θ	10	20	30	40	45	50	60	70	80	90
2·0	0·9983	0·9935	0·9864	0·9783	0·9744	0·9712	0·9683	0·9734	0·9883	1·0
1·5	0·9975	0·9906	0·9798	0·9666	0·9598	0·9534	0·9441	0·9453	0·9676	1·0
1·4	0·9974	0·9897	0·9779	0·9632	0·9555	0·9481	0·9366	0·9356	0·9582	1·0
1·3	0·9971	0·9888	0·9757	0·9593	0·9506	0·9420	0·9277	0·9237	0·9452	1·0
1·2	0·9969	0·9877	0·9733	0·9549	0·9449	0·9349	0·9172	0·9091	0·9273	1·0
1·1	0·9966	0·9865	0·9704	0·9498	0·9383	0·9266	0·9047	0·8913	0·9031	1·0
1·0	0·9962	0·9850	0·9672	0·9438	0·9306	0·9169	0·8900	0·8697	0·8715	1·0
0·9	0·9958	0·9834	0·9633	0·9368	0·9222	0·9055	0·8726	0·8437	0·8318	0·9
0·8	0·9953	0·9814	0·9589	0·9286	0·9110	0·8921	0·8521	0·8128	0·7841	0·8
0·7	0·9947	0·9791	0·9536	0·9189	0·8985	0·8763	0·8278	0·7766	0·7286	0·7
0·6	0·9941	0·9764	0·9473	0·9074	0·8837	0·8516	0·7994	0·7345	0·6660	0·6
0·5	0·9932	0·9730	0·9397	0·8936	0·8660	0·8355	0·7660	0·6862	0·5972	0·5
0·4	0·9922	0·9690	0·9305	0·8771	0·8449	0·8092	0·7272	0·6316	0·5225	0·4
0·3	0·9910	0·9640	0·9193	0·8572	0·8197	0·7780	0·6821	0·5702	0·4427	0·3
0·2	0·9894	0·9578	0·9055	0·8329	0·7892	0·7407	0·6300	0·5018	0·3578	0·2
0·1	0·9874	0·9499	0·8881	0·8031	0·7523	0·6962	0·5697	0·4260	0·2682	0·1
0·0	0·9848	0·9397	0·8660	0·7660	0·7071	0·6428	0·5000	0·3420	0·1736	0·0
−0·1	0·9813	0·9261	0·8374	0·7194	0·6512	0·5779	0·4190	0·2490	0·0741	−0·1
−0·2	0·9764	0·9075	0·7991	0·6593	0·5807	0·4977	0·3236	0·1451	−0·0311	−0·2
−0·3	0·9692	0·8809	0·7457	0·5786	0·4880	0·3952	0·2085	0·0275	−0·1429	−0·3
−0·4	0·9583	0·8410	0·6645	0·4596	0·3551	0·2528	0·0612	−0·1103	−0·2632	−0·4
−0·5	0·9397	0·7660	0·5000	0·1736	0·0000	−0·0481	−0·1736	−0·2868	−0·3961	−0·5
−0·6	0·9015	—	—	—	—	—	—	—	−0·5542	−0·6
−0·7	—	—	—	—	—	—	—	—	—	−0·7
−0·8	—	—	—	—	—	—	—	—	—	−0·8
−0·9	—	—	—	—	—	—	—	—	—	−0·9
−1·0	—	—	—	—	—	—	—	—	—	−1·0
$\cos\phi_0$	0·7740	0·5660	0·3456	0·1162	0·0000	−0·1162	−0·3456	−0·5660	−0·7740	−1·0
$\cos\phi_0'$	−0·9946	−0·9750	−0·9416	−0·8940	−0·8660	−0·8361	−0·7745	−0·7224	−0·7128	−1·0

Critical values (tables 4 and 5)

For any orientation, θ, of the polar axis of a prolate ellipsoid relative to the field, precise calculations may be made, using the equations (2·12) to (2·18), of the critical field, h_0, in passing through which a discontinuous change in the equilibrium direction of the magnetization occurs, and of the angles, ϕ_0 and ϕ_0', between the magnetization, I_0, and the

(positive) field direction at the beginning and end of the 'jump'. Values of h_0, ϕ_0 and ϕ_0', and the corresponding values of the reduced resolved magnetization, $\cos\phi_0$ and $\cos\phi_0'$, and of the magnitude of the jump, are given in table 4. The variation of ϕ_0 and ϕ_0' (and of the corresponding cosines) with θ is somewhat complex (as will be apparent from figures 4, 5 and 6), and without a very large number of entries, a table at equal intervals in θ over the range $0 \leqslant \theta \leqslant 90°$ (such as is provided incidentally in tables 2 and 3) does not bring out clearly essential characteristics of the behaviour. Entries have therefore been made for values of θ which are symmetrical about $\theta = 45°$, but at intervals ranging from 1 to 10°, according to the character of the variation. If h_{01}, ϕ_{01}, and $\cos\phi_{01}$ refer to an angle θ_1, then for $\theta_2 = 90 - \theta_1$, the following relations hold: $h_{02} = h_{01}$, $\phi_{02} = 90 + \phi_{01}$, $\cos\phi_{02} = -\cos\phi_{01}$.

TABLE 4. CRITICAL FIELD, h_0, CRITICAL ANGLE OF MAGNETIZATION, ϕ_0, AND RELATED QUANTITIES AS DEPENDENT ON ORIENTATION, θ, OF PROLATE SPHEROID

ϕ_0 and ϕ_0' are the initial and final angles made by the magnetization vector, I_0, with the positive direction of the field, as h, decreasing from a positive value greater than $|h_0|$, passes through the value h_0. The resolved value of the magnetization in the positive direction of the field is given by $I_0 \cos\phi$. The jump in the resolved magnetization at h_0 is given by $I_0(\cos\phi_0' - \cos\phi_0)$.

$h = H/(N_b - N_a)\, I_0$, where N_a and N_b are the demagnetization coefficients along the polar and equatorial axes respectively.

θ	$-h_0$	ϕ_0	$\cos\phi_0$	$\phi_0' - 180$	$-\cos\phi_0'$	$\cos\phi_0' - \cos\phi_0$
0	1·00000	0·000	1·00000	0·000	1·00000	−2·00000
1	0·90707	15·542	0·96343	0·524	0·99996	−1·96339
2	0·85929	20·100	0·93919	1·076	0·99982	−1·93901
4	0·79237	26·391	0·89578	2·231	0·99924	−1·89502
6	0·74370	31·264	0·85479	3·440	0·99820	−1·85299
8	0·70531	35·471	0·81441	4·689	0·99665	−1·81106
10	0·67381	39·287	0·77404	5·971	0·99458	−1·76862
12	0·64733	42·829	0·73339	7·279	0·99194	−1·72533
14	0·62475	46·186	0·69232	8·610	0·98873	−1·68105
16	0·60527	49·402	0·65075	9·958	0·98493	−1·63568
18	0·58838	52·508	0·60865	11·322	0·98054	−1·58920
20	0·57365	55·526	0·56603	12·697	0·97555	−1·54158
30	0·52402	69·784	0·34557	19·671	0·94164	−1·28721
40	0·50255	83·326	0·11623	26·618	0·89402	−1·01025
45	0·50000	90·000	0·00000	30·000	0·86603	−0·86603
50	0·50255	96·674	−0·11623	33·269	0·83610	−0·71987
60	0·52402	110·216	−0·34557	39·238	0·77452	−0·42896
70	0·57365	124·474	−0·56603	43·749	0·72237	−0·15634
72	0·58838	127·492	−0·60865	44·337	0·71524	−0·10659
74	0·60527	130·598	−0·65075	44·762	0·71004	−0·05929
76	0·62475	133·814	−0·69232	44·982	0·70733	−0·01501
78	0·64733	137·171	−0·73339	44·936	0·70789	+0·02550
80	0·67381	140·713	−0·77404	44·534	0·71283	+0·06121
82	0·70531	144·529	−0·81441	43·630	0·72380	+0·09061
84	0·74370	148·736	−0·85479	41·968	0·74352	+0·11126
86	0·79237	153·609	−0·89578	39·013	0·77700	+0·11878
88	0·85929	159·900	−0·93919	33·277	0·83603	+0·10316
89	0·90707	164·458	−0·96343	27·609	0·88614	+0·07730
90	1·00000	180·000	−1·00000	0·000	1·00000	0·00000

In these, as in the previous tables, the values listed are those appropriate for h decreasing from a positive value greater than $+|h_0|$ for each particular orientation angle, θ. (For increasing h, the values can be written down from those listed as described in connexion with tables 2 and 3.) An interesting feature of the results is that although, for decreasing h, ϕ_0' is always greater than ϕ_0, the jump, as measured by $\cos\phi_0'-\cos\phi_0$, is not negative (i.e. in the direction corresponding to an increasing negative resolved magnetization) over the whole range. Table 4 shows that the jump becomes positive at a value of θ between 76 and 78°, then at first increases, and finally decreases to zero for $\theta = 90°$. The value of θ at which the change in angle from ϕ_0 to ϕ_0' corresponds to a zero change in the resolved magnetization, that is to equality of $\cos\phi_0$ and $\cos\phi_0'$, can be determined without difficulty from the basic equations (2·10) and (2·11), noting that $\phi_0' = 2\pi-\phi_0$, and that ϕ_0 corresponds to a 'point of inflexion' (a zero value in (2·11)) and ϕ_0' to a minimum in the energy (a positive value in (2·11)). Manipulation of the equations shows that, in the ranges $0<\theta<\frac{1}{2}\pi$, $0<\phi_0<\pi$, the only value of ϕ_0 for which the requirements are satisfied is $3\pi/4$. The corresponding value of θ is $\frac{1}{2}\pi+\frac{1}{2}\tan^{-1}2$, and of h_0, $-(2/5)^{\frac{1}{2}}$. Numerical values are given in table 5.

TABLE 5. CRITICAL ORIENTATION, θ, CRITICAL ANGLE OF MAGNETIZATION, ϕ_0, AND RELATED QUANTITIES AS DEPENDENT ON CRITICAL FIELD, h_0, FOR PROLATE SPHEROID

$-h_0$	θ	ϕ_0	$\cos\phi_0$	$\phi_0'-180$	$-\cos\phi_0'$	$\cos\phi_0'-\cos\phi_0$
1·0	0·000	0·000	1·00000	0·000	1·00000	−2·00000
0·9	1·128	16·238	0·96011	0·593	0·99995	−1·96006
0·8	3·732	25·659	0·90139	2·073	0·99935	−1·90074
0·7	8·312	36·087	0·80812	4·887	0·99636	−1·80449
0·6	16·595	50·336	0·63829	10·362	0·98369	−1·62198
0·5	45·000	90·000	0·00000	30·000	0·86603	−0·86603
0·6	73·405	129·664	−0·63829	44·655	0·71136	−0·07307
0·7	81·688	143·913	−0·80812	43·813	0·72160	+0·08652
0·8	86·268	154·341	−0·90139	38·467	0·78296	+0·11843
0·9	88·872	163·762	−0·96011	28·558	0·87834	+0·08177
1·0	90·000	180·000	−1·00000	0·000	1·00000	0·00000
0·63245	76·717	135·000	−0·70711	45·000	0·70711	0·00000

Random orientation (table 6)

The results for the variation with field (H descending) of the resolved magnetization for an assembly of similar prolate spheroids with polar axes orientated at random are shown in table 6.

The character of the variation shown by the table is most readily appreciated from figure 7. Two particular points may be noticed. The first is that the 'cross-over', due to the positive jumps when θ is in the range $76{\cdot}72<\theta<90°$, is negligible in the mean curves, being apparent in the table only for $h=-0{\cdot}9$, for which the value for $|\overline{\cos\phi}|$ is just greater than that for $h=+0{\cdot}9$ (by $0{\cdot}0002_3$, according to the very carefully checked 5-place computations). The second point is that the coercivity, $|h_c|$, is numerically less than the smallest value of $|h|$ for which jumps occur. Thus, for descending h, the discontinuous changes in magnetization occur only in the region in which the resolved magnetization is negative.

With the tabular intervals of table 6, it is not possible to make an accurate estimate of h_c. An approximate estimate is $h_c=-0{\cdot}479$. (Owing to the discontinuous change in the first

derivative of the function at $h = -0{\cdot}5$, inverse interpolation can be carried out only by using the backward Gregory-Newton formula, or some equivalent, and the convergence is very slow. Using differences up to the 3rd, 4th, 5th and 6th, the calculated values of $-h_c$ are 0·4772, 0·4779, 0·4783 and 0·4786 respectively. For a more accurate estimate, it would be necessary to calculate $\overline{\cos\phi}$ at smaller intervals in h, and the labour entailed in this would be out of all proportion to the theoretical or practical usefulness of a more accurate estimate of h_c).

TABLE 6. VARIATION OF MEAN RESOLVED MAGNETIZATION WITH FIELD (H DESCENDING) FOR PROLATE SPHEROIDS ORIENTATED AT RANDOM

h	$\overline{\cos\phi}$	$-h$	$\overline{\cos\phi}$
2·0	0·9809	0·0	0·5000
1·5	0·9646	0·1	0·4293
1·4	0·9588	0·2	0·3488
1·3	0·9516	0·3	0·2548
1·2	0·9422	0·4	0·1380
1·1	0·9298	0·5	−0·0526
1·0	0·9130	0·6	−0·6978
0·9	0·8907	0·7	−0·8145
0·8	0·8641	0·8	−0·8624
0·7	0·8335	0·9	−0·8909
0·6	0·7991	1·0	−0·9130
0·5	0·7608	1·1	−0·9298
0·4	0·7184	1·2	−0·9422
0·3	0·6716	1·3	−0·9516
0·2	0·6201	1·4	−0·9588
0·1	0·5632	1·5	−0·9646

Magnetization curves (figures 6 and 7)

Illustrative magnetization curves, drawn from table 3, are shown in figure 6 for $\theta = 0$, 10, 45, 80 and 90°. The $\cos\phi_0$ and $\cos\phi_0'$ curves are also shown, drawn from tables 4 and 5. The figure shows clearly the remarkable change in the character of the magnetization curves for change in orientation, from the rectangular hysteresis loop for $\theta = 0$, to the straight line, non-hysteretic, curve for $\theta = 90°$. The magnitude of the jumps is given by the difference between $\cos\phi_0$ and $\cos\phi_0'$, the upper half of the former and the lower half of the latter curve applying to the range $0 \leqslant \theta \leqslant 45°$. The values $\theta = 10°$ and $\theta = 80°$ have been chosen so as to bring out the inter-relations between the curves for two angles θ_1 and θ_2 such that $\theta_2 = 90° - \theta_1$.

The magnetization curve corresponding to the mean resolved magnetization of an assembly of similar spheroids, orientated at random, is shown in figure 7, the full curves referring to prolate spheroids, discussed above, and the broken curves to oblate spheroids (see § 5).

The initial curve for the prolate spheroid assembly shown in figure 7 is the mean of the descending and ascending branches of the hysteresis curve. It corresponds to the course of the magnetization for an assembly initially completely demagnetized, in the sense that, for spheroids orientated at an angle θ to the positive field direction, the magnetization vector for one half the spheroids is along the polar axis in one sense ($\phi = \theta$) and for the other half in the opposite sense ($\phi = \pi + \theta$).

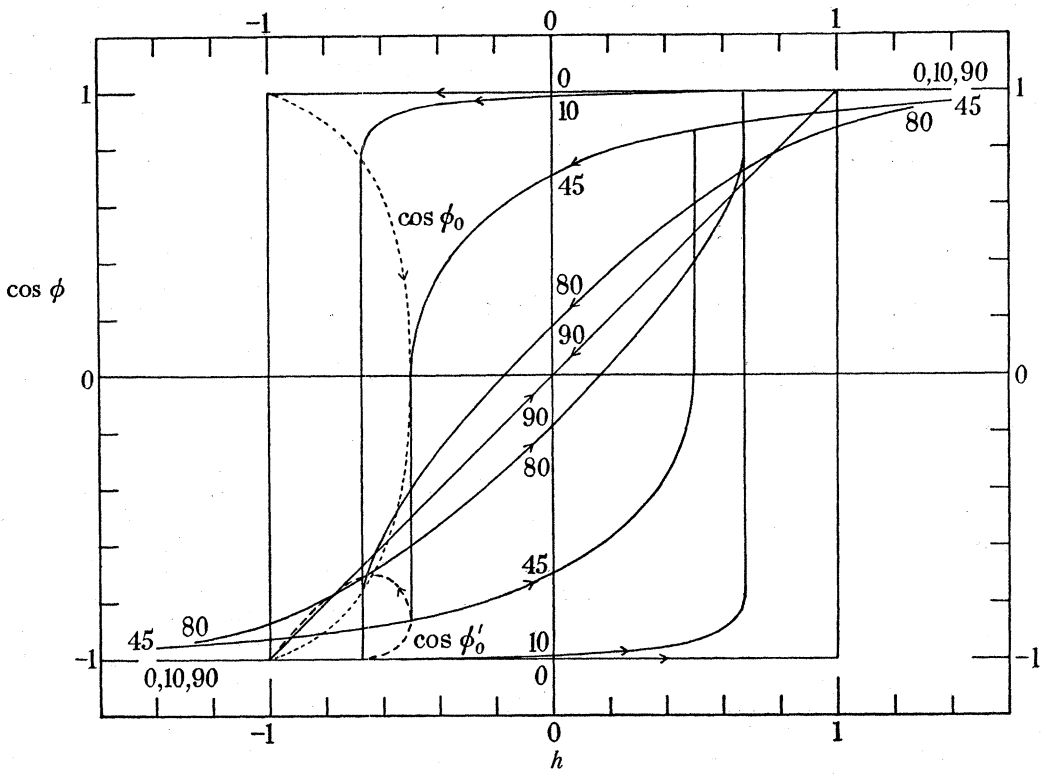

FIGURE 6. Magnetization curves for prolate spheroids. The resolved magnetization in the positive field direction is given by $I_0 \cos \phi$, where I_0 is the saturation magnetization. The field, H, is given by $H = (N_b - N_a)\, I_0 h$, where N_a and N_b are the demagnetization coefficients along the polar and equatorial axes. The angle, θ, between the polar axis and the direction of the field, is shown, in degrees, by the numbers on the curves. The dotted curves give $\cos \phi_0$ and $\cos \phi_0'$, where ϕ_0 and ϕ_0' are the angles made with the positive field direction by the magnetization vector at the beginning and end of the discontinuous change at the critical value, h_0, of the field.

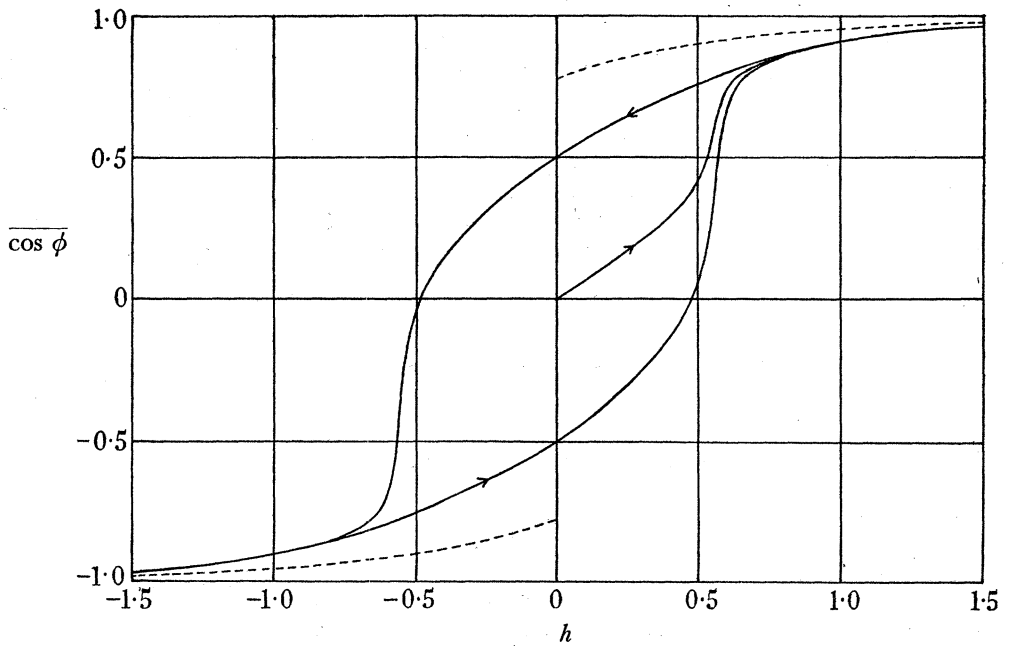

FIGURE 7. Magnetization curves for prolate (full curves) and oblate (broken curves) spheroids orientated at random. The curves refer to similar prolate (or oblate) spheroids orientated at random. $\overline{\cos \phi}$ is proportional to the mean resolved magnetization per spheroid in the positive field direction, or to the resultant magnetization in this direction of the assembly. $H = (|N_a - N_b|)\, I_0 h$.

Among various characterizing magnitudes for the random assembly of prolate ellipsoids are the following:

Coercivity: $$H_c = 0{\cdot}479(N_b - N_a)\, I_0. \tag{4·1}$$

Initial susceptibility: $$\kappa_0 = \frac{2}{3}\frac{1}{N_b - N_a}. \tag{4·2}$$

Remanent intensity of magnetization: $$I_r = \tfrac{1}{2} I_0. \tag{4·3}$$

From the above, the coercivity and initial susceptibility are related by

$$H_c = 0{\cdot}319(I_0/\kappa_0). \tag{4·4}$$

In these equations, I_0 is to be understood as the saturation magnetization (per unit volume) of the material of the ellipsoidal ferromagnetic particles themselves, and, for each particle, H is the field originating externally to itself and acting on the particle. If I_H is the mean resolved value of I_0 in the field direction, κ_0 is the initial value of (dI_H/dH). For a specimen of material consisting of ferromagnetic particles dispersed in a non-ferromagnetic (or less ferromagnetic) matrix, the relation between H, as used here, and the actual field applied to the specimen would require careful consideration, in the usual manner, in each particular case. While it might be difficult to derive, with certainty, the values of κ_0, to which the above equations refer, from experimentally measured susceptibilities, the difficulty does not arise with H_c, which, with a non-ferromagnetic matrix at least, would be identifiable with the coercivity as ordinarily measured.

Conversion factors

The field, H, is related to the reduced field, h, as used in the tables and figures, by

$$H = (N_b - N_a)\, I_0 h, \tag{4·5}$$

as given in (2·8). Here N_a and N_b are the demagnetization coefficients for magnetization along the polar and an equatorial axis respectively. For a prolate spheroid, $N_b > N_a$. (For the oblate spheroid, it is appropriate to put $h = (N_a - N_b)\, I_0 h$, see § 5.) Critical values of the reduced field fall in the range $\frac{1}{2} \leqslant |h_0| \leqslant 1$, and in order that the significance of these values may be appreciated, it is appropriate to give a short table (table 7) of the values of $N_b - N_a$ for ellipsoids of revolution of different dimensional ratios, $m (m = a/b)$. It may be noted that, in view of (2·2) and (2·6), for a prolate spheroid $(m > 1)$,

$$N_b - N_a = 2\pi - \tfrac{3}{2} N_a, \tag{4·6}$$

and for an oblate spheroid $(m < 1)$,

$$N_a - N_b = \tfrac{3}{2} N_a - 2\pi. \tag{4·7}$$

Numerical values are readily obtained by the use of the detailed tables of values of D_a $(D_a = N_a/4\pi)$ for ellipsoids of revolution given by Stoner (1945).

The general character of the dependence of $|N_a - N_b|$ on the dimensional ratio, m, is perhaps most readily appreciated from the logarithmic plot shown in figure 8.

The maximum value of the reduced critical field, and of the coercivity, for prolate spheroids is unity. The corresponding value of H, for prolate spheroids of given dimensional

ratio, is obtained by multiplying the numbers in table 7 by I_0, for which a representative order of magnitude is 10^3. (For iron, $I_0 = 1{\cdot}7 \times 10^3$.) It will be apparent that for elongated ellipsoids, values of H_0 of several thousand oersteds are possible, and that even for prolate

TABLE 7. VALUES OF $|N_a - N_b|$ FOR ELLIPSOIDS OF REVOLUTION

$m = a/b$. a, polar axis; b, equatorial axis. N_a, N_b, demagnetization coefficients for magnetization along polar, equatorial axis.

oblate spheroid		prolate spheroid			
m	$N_a - N_b$	m	$N_b - N_a$	m	$N_b - N_a$
0·0	12·566	1·0	0·000	2	3·012
0·1	9·943	1·1	0·472	3	4·234
0·2	7·863	1·2	0·890	4	4·862
0·3	6·183	1·3	1·261	5	5·231
0·4	4·803	1·4	1·593	6	5·468
0·5	3·654	1·5	1·892	10	5·901
0·6	2·686	1·6	2·161	20	6·156
0·7	1·861	1·7	2·404	30	6·218
0·8	1·152	1·8	2·625	40	6·243
0·9	0·537	1·9	2·827	50	6·256
1·0	0·000	2·0	3·012	∞	6·283

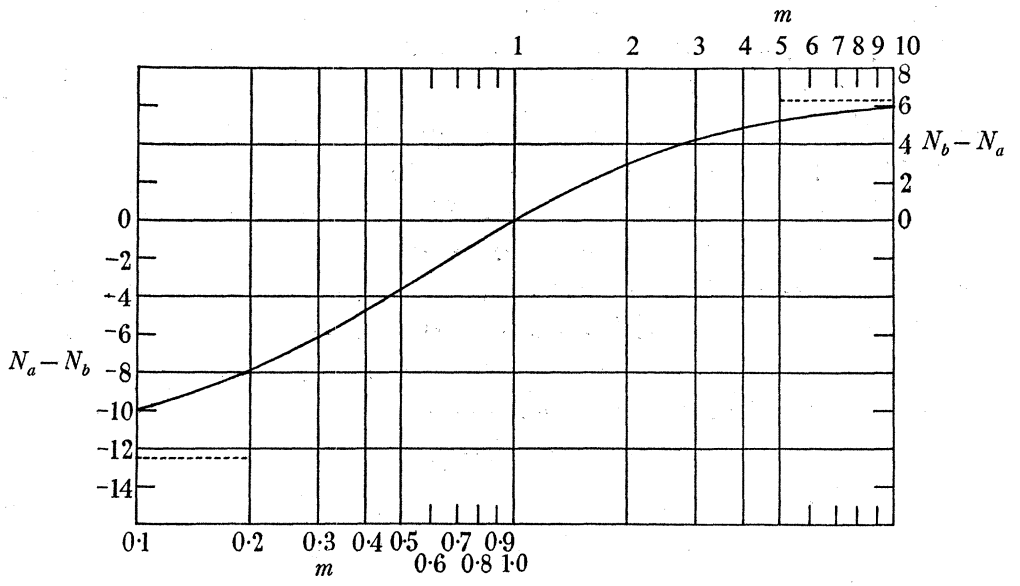

FIGURE 8. Dependence of the difference, $|N_a - N_b|$, between the principal demagnetization coefficients of ellipsoids of revolution on the dimensional ratio, m. $m = a/b$. a, polar axis; b, equatorial axis. The m scale is logarithmic.

spheroids which are nearly spherical in form (say $m = 1{\cdot}1$ or $1{\cdot}2$), values of several hundred oersteds are possible. It will be shown in the next section that for oblate spheroids discontinuities in magnetization occur only when the field passes through zero and that the coercivity is also zero. For the general ellipsoid, coercivity values intermediate between those for prolate and oblate spheroids are to be expected.

5. The oblate spheroid and the general ellipsoid

(i) *The oblate spheroid*

The relation between the character of the variation with field of the magnetization for an oblate, and for a prolate spheroid becomes fairly clear on consideration of diagrams similar to figure 1. Denoting by θ' the angle between the polar axis, a, of the oblate spheroid, and the positive direction of the field, it is at once apparent that if I_0 is restricted to the H, a plane, the directions of I_0 corresponding to stationary energy values will be the same as those for a prolate spheroid for which the orientation angle, θ, is equal to $\theta' \pm \frac{1}{2}\pi$. The directions correspond to true energy minima, however, only when the resolved component of I_0 is in the direction of the field. When the field has been reduced from a positive value to zero, I_0 lies along an equatorial axis in the H, a plane in a sense such that the angle made with the positive direction of the field is less than 90°. When H is zero, the energy is the same for all directions of I_0 in the equatorial plane, and as H passes through zero to a negative value, I_0 moves round in the equatorial plane to a direction again in the H, a plane at 180° to the original direction, and making an angle of less than 90° with the negative value of H. Discontinuities in the resolved magnetization thus occur only at $H = 0$, when they correspond to a reversal in sign of $\cos\phi'$, and there is no hysteresis. The treatment of the general ellipsoid outlined below provides a formal proof that the direction of I_0 moves out of the H, a plane for an oblate spheroid only as H passes through the value zero, and also that for a prolate spheroid the equilibrium direction of I_0 is always in the H, a plane. Acceptance of these conclusions as 'physically obvious', however, makes possible a very simple formal treatment of the interrelation between the magnetic behaviour of prolate and oblate spheroids, leading to a statement which enables numerical results for the oblate spheroid to be readily obtained from the tables already given for the prolate spheroid.

The equations for the dependence of energy on direction of magnetization, when the magnetization vector is in the plane defined by the field and the polar axis of an ellipsoid of revolution are given in § 2. In reducing these equations to non-dimensional form it is appropriate for the oblate spheroid ($a<b$, $N_a>N_b$) to divide by the positive quantity $(N_a-N_b)\,I_0$. Denoting the associated values of the orientation angle, magnetization angle, and reduced field for the oblate spheroid by θ', ϕ', and h', where

$$h' = H/(N_a-N_b)\,I_0, \tag{5·1}$$

directions of magnetization corresponding to stationary energy values are determined by

$$-\tfrac{1}{2}\sin 2(\phi'-\theta') + h'\sin\phi' = 0. \tag{5·2}$$

Values of ϕ' given by this equation correspond to energy minima, points of inflexion, and maxima, for

$$-\cos 2(\phi'-\theta') + h'\cos\phi' \gtreqless 0. \tag{5·3}$$

Comparison of (5·2) and (5·3) with (2·10) and (2·11) then shows that, provided I_0 is restricted to the H, a plane, the relation between the sets of associated values for oblate and prolate spheroids may be stated as follows:

> If the set of values θ, ϕ, h corresponds to a stationary energy value (minimum, inflexion, maximum) for the prolate spheroid, then the set of values θ', ϕ', h' corresponds to a stationary energy value (minimum, inflexion, maximum) for the oblate spheroid if
>
> $$h' = h, \quad \theta' = \tfrac{1}{2}\pi - \theta, \quad \phi' = -\phi. \tag{5·4}$$

Using (5·4), the values of ϕ' for energy minima for the oblate spheroid can therefore be immediately obtained from table 2, but, owing to the rotation of I_0 in the equatorial plane as H passes through the value zero, the relevant h range is restricted to that for which $h \geqslant 0$. Thus, for $\theta' = 20°$, the values of ϕ' are the negative of the values of ϕ given in table 2 under $\theta = 70°$ over the h range from $+2$ to 0. The values of $\cos\phi'$ are similarly obtained from table 3, but, since $\cos(-\phi) = \cos\phi$, without change of sign. It may be noted that, as far as maxima and minima in the H, a plane are concerned, figures 4 and 5 become applicable to oblate spheroids if the signs of the values given on the h contours are reversed, and the full and broken curves are taken to refer to energy maxima and minima respectively.

It is not possible to derive values of the mean resolved magnetization of oblate spheroids orientated at random from those for prolate spheroids, but the numerical integrations present no special difficulty. The results are given in table 8, and are shown graphically in figure 7.

TABLE 8. VARIATION OF MEAN RESOLVED MAGNETIZATION WITH FIELD FOR OBLATE SPHEROIDS ORIENTATED AT RANDOM

$\overline{\cos\phi}$ is proportional to the mean resolved magnetization per spheroid in the positive field direction. $H = (N_a - N_b)\, I_0 h$. For $h = -|h|$, $\cos\phi$ is the negative of the value for $h = +|h|$. In passing through $h = 0$ from positive to negative, $\overline{\cos\phi}$ changes from $+\frac{1}{4}\pi$ to $-\frac{1}{4}\pi$.

h	$\overline{\cos\phi}$	h	$\overline{\cos\phi}$
2·0	0·9858	0·5	0·9000
		0·4	0·8835
1·5	0·9762	0·3	0·8643
1·4	0·9732	0·2	0·8420
1·3	0·9696	0·1	0·8160
1·2	0·9652		
1·1	0·9600	$0{\cdot}0_+$	0·7854
1·0	0·9537	$0{\cdot}0_-$	−0·7854
0·9	0·9461		
0·8	0·9371	−0·1	−0·8160
0·7	0·9266	−0·2	−0·8420
0·6	0·9143	−0·3	−0·8643

(ii) *The general ellipsoid*

Although a detailed numerical investigation of the magnetic behaviour of an assembly of ferromagnetic particles of general ellipsoidal form would hardly yield results of a value commensurate with the labour entailed, an examination of the relations for the general ellipsoid is necessary for two reasons. First, formal confirmation is required for the conclusions drawn as to the stability of the state of an ellipsoid of revolution in which the magnetization vector lies in the plane defined by the field and the polar axis; and secondly, it would be unwarrantable to base general conclusions about the behaviour of assemblies of ferromagnetic particles solely on the examination of such specialized forms as strict ellipsoids of revolution.

Basic equations. In dealing with the general ellipsoid it is convenient to take as co-ordinate axes the principal axes of the ellipsoid. The directions, and also the semi-axes, are denoted by a_1, a_2, a_3 and the corresponding demagnetization coefficients by N_1, N_2, N_3. The angles made by I_0 with the axes are denoted by α_1, α_2, α_3 and those made by H by $\beta_1, \beta_2, \beta_3$. To avoid

undue complication the cases to be examined will be limited to those in which H lies in a principal plane, say the a_1, a_2 plane, so that

$$\cos\beta_3 = 0, \quad \cos\beta_2 = \sin\beta_1. \tag{5·5}$$

By use of the relation

$$\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1, \tag{5·6}$$

the problem is reduced to one in two variables, and since particular interest attaches to the question of whether I_0 leaves the a_1, a_2 plane, these variables are conveniently taken as α_1, α_3. The energy equation, omitting constant terms, can then be written

$$E = \tfrac{1}{2}I_0^2\{(N_1 - N_2)\cos^2\alpha_1 + (N_3 - N_2)\cos^2\alpha_3\} - HI_0\{\cos\alpha_1\cos\beta_1 + (1-\cos^2\alpha_1-\cos^2\alpha_3)^{\frac{1}{2}}\sin\beta_1\}. \tag{5·7}$$

Writing

$$\frac{\partial E}{\partial\alpha_1} = f_1, \quad \frac{\partial E}{\partial\alpha_3} = f_3, \quad \frac{\partial^2 E}{\partial\alpha_1^2} = f_{11}, \quad \frac{\partial^2 E}{\partial\alpha_3^2} = f_{33}, \quad \frac{\partial^2 E}{\partial\alpha_1\partial\alpha_3} = f_{13},$$

the conditions for an energy minimum are

$$f_1 = 0, \quad f_3 = 0, \tag{5·8}$$

$$f_{11}f_{33} - f_{13}^2 > 0, \tag{5·9}$$

$$f_{11} > 0 \quad (\text{or } f_{33} > 0). \tag{5·10}$$

For a stationary value with respect to α_3,

$$f_3 = \sin\alpha_3\cos\alpha_3\{(N_2 - N_3)I_0^2 - HI_0(1-\cos^2\alpha_1-\cos^2\alpha_3)^{-\frac{1}{2}}\sin\beta_1\} = 0. \tag{5·11}$$

Three cases in which this equation is satisfied may be distinguished.

(i) $\cos\alpha_3 = 0$ (i.e. I_0 in the a_1, a_2 plane).

With this value substituted in the general expressions for the derivatives, $f_{13} = 0$, and the conditions for an energy minimum are

$$f_1 = \tfrac{1}{2}(N_2 - N_1)\sin 2\alpha_1 + HI_0\sin(\alpha_1 - \beta_1) = 0, \tag{5·12}$$

$$f_{11} = (N_2 - N_1)\cos 2\alpha_1 + HI_0\cos(\alpha_1 - \beta_1) > 0, \tag{5·13}$$

$$f_{33} = (N_3 - N_2)I_0^2 + HI_0(\sin\beta_1/\sin\alpha_1) > 0. \tag{5·14}$$

The content of (5·12) and (5·13) is essentially the same as that of (2·10) and (2·11) (for the prolate spheroid) and of (5·2) and (5·3) (for the oblate spheroid). The equations give the conditions for an energy minimum with respect to change of direction of I_0 in the a_1, a_2 plane. Such a minimum is a true minimum only if (5·14) is also satisfied. The discussion of this case is taken up further below.

(ii) $\sin\alpha_3 = 0$ (i.e. I_0 normal to the a_1, a_2 plane).

This solution is associated with a true energy minimum only if N_3 is smaller than N_1 and N_2 (a_3 greater than a_1 and a_2) and for the particular field value, $H = 0$. (It is unnecessary to enter here into the details which a rigorous treatment would require.)

(iii) $(N_2 - N_3)I_0^2 = HI_0(1-\cos^2\alpha_1-\cos^2\alpha_3)^{-\frac{1}{2}}\sin\beta_1$.

When H decreases from a large positive value, this solution is relevant as being associated with a true energy minimum only when I_0 leaves the a_1, a_2 plane while H is still positive to pass through a position normal to the plane (case (ii)) when H is zero. The equation then

determines, in conjunction with the associated equations for f_1, f_{11} and f_{33}, the equilibrium direction of I_0 (dependent on H and β_1) when it is neither in, nor at right angles to, the principal plane a_1, a_2. The resulting equations are very complicated and have not been investigated in detail, but there is nothing to suggest that the decrease in $\cos\alpha_1$ over this range is other than monotonic. It may be noted, moreover, that although I_0 is not in the a_1, a_2 plane, in this case, as well as in case (ii), for any assembly of ellipsoids, the normal component of the magnetization remains equal to zero, for there are alternative solutions of the equations corresponding to values of $\cos\alpha_3$ which are equal but of opposite sign.

Magnetization in principal plane. The outline just given is sufficient to indicate the character of the sequence of equilibrium directions of the magnetization vector when it moves out of a principal plane in which the field is applied, and also to show how the course of such changes could, if required, be followed in detail. In order to bring out the main points more clearly, and in such a way as to link up with the treatment given of ellipsoids of revolution, it is convenient to re-express some of the results in terms of the angles θ, ϕ and ψ previously used (see figure 1), and which are appropriate when the magnetization is in the same plane as H and a principal axis. The reference axis of the ellipsoid is taken as a_1 (which may be greater than, less than, or intermediate between a_1 and a_2), and the field is taken to lie in the a_1, a_2 plane. The relations between the angles θ, ϕ and ψ and the angles used in (5·5) to (5·14) are

$$\alpha_1 = \psi = \phi - \theta, \quad \beta_1 = -\theta, \quad \text{giving} \quad \alpha_1 - \beta_1 = \psi + \theta = \phi. \tag{5·15}$$

The angle θ gives the orientation of the a_1 axis with respect to the field, and ϕ gives the angle between I_0 and H, so that, as before, $\cos\phi$ gives the resolved magnetization in the (positive) field direction. There are now two sets of critical values to be determined. The first is that associated with a discontinuous change of $\cos\phi$ when the magnetization is restricted to the a_1, a_2 plane, and is derivable from (5·12) and (5·13), which give, as the basic relation (cf. (2·12) to (2·17)),

$$\tan\psi_0 = \tan^{\frac{1}{3}}\theta = t. \tag{5·16}$$

The second is that corresponding to a deviation of I_0 from the a_1, a_2 plane, and is derivable from (5·12) and (5·14), which give as the basic relation,

$$\tan\psi_0^* = \frac{N_1 - N_3}{N_3 - N_2}\tan\theta = \frac{1}{r}\tan\theta. \tag{5·17}$$

The significance of these relations is most readily appreciated from a diagram, and the critical values ϕ_0^* ($\phi_0^* = \psi_0^* + \theta$) are accordingly plotted in figure 9 for a series of values of r, and also the critical values ϕ_0 (cf. figures 4 and 5).

The behaviour to be considered in connexion with figure 9 is that for H decreasing from a large positive value. For $N_1 < N_2$, for any particular angle θ, the sequence of ϕ values, so long as I_0 is restricted to the a_1, a_2 plane, is that shown by the constant field contours of figures 4 and 5, and numerically in table 2, for the prolate spheroid. As H decreases, ϕ increases from zero, and the representative point moves vertically upwards from the line $\phi = 0$ (corresponding to $H = +\infty$), until it reaches the ϕ_0 curve, after crossing the line $\phi = \theta$, corresponding to $H = 0$. If the r value for the ellipsoid is such that the ϕ_0^* curve lies above the ϕ_0 curve, the course of the magnetization change is the same as that for a prolate spheroid with $N_a = N_1$, $N_b = N_2$. If, however, the ϕ_0^* curve is below the ϕ_0 curve, the value of H_0, at which I_0 leaves the a_1, a_2 plane is algebraically greater than the critical field H_0 for the

prolate spheroid (if H_0 is negative, it is numerically smaller than H_0), and the effective width of the hysteresis curve is correspondingly reduced.

For $N_2 < N_1$, the representative point moves downwards from the top of the diagram, and ϕ (right-hand scale) increases negatively from zero as H decreases from $+\infty$, the limiting curve (not attained for the oblate spheroid) again being the ϕ_0 curve. The constant field contours are as in figures 4 and 5, but with the signs reversed, and with the broken curves corresponding to energy minima.

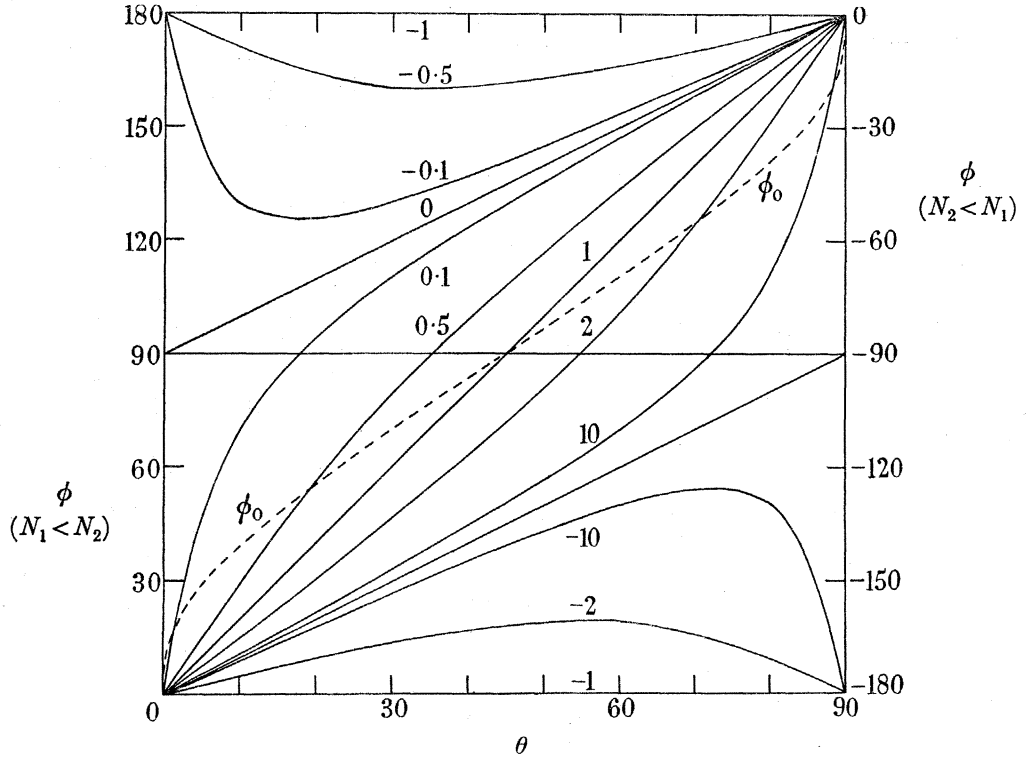

FIGURE 9. Critical angles for magnetization in principal plane of general ellipsoid. The field is in the a_1, a_2 plane. θ, angle between a_1 and positive direction of H. ϕ, angle between I_0 and positive direction of H. ϕ_0, angle at which $I_0 \cos\phi$ changes discontinuously, for H decreasing, when I_0 is restricted to a_1, a_2 plane. The curves, other than that labelled ϕ_0, give the values of ϕ_0^*, the angle at which I_0 leaves the a_1, a_2 plane. N_1, N_2, N_3, demagnetization coefficients along a_1, a_2, a_3 axes. The numbers on the ϕ_0^* curves give the values of r, where $r = (N_3 - N_2)/(N_1 - N_3)$.

The magnetization vector may remain always in the principal plane, a_1, a_2. Thus for $N_1 < N_2 < N_3$, r necessarily lies between -1 and 0, and the corresponding ϕ_0^* curves all lie above the ϕ_0 curve. The characteristics in the six possible cases are summarized in table 9.

It may be noted that the ϕ_0^* curves cut the ϕ_0 curve only when r is positive, the orientation angle θ, at which the intersection occurs, being given, from (5·16) and (5·17), by

$$\tan\theta = r^{\frac{3}{2}}. \tag{5·18}$$

With θ in the range 0 to $\frac{1}{2}\pi$, and with r positive, the full hysteretic behaviour characteristic of the state in which I_0 is restricted to the a_1, a_2 plane is still shown for certain ranges of θ: with $N_1 < N_2$, for $\theta > \tan^{-1} r^{\frac{3}{2}}$, and with $N_2 < N_1$, for $\theta < \tan^{-1} r^{\frac{3}{2}}$. The difference here between the cases (2) and (2′) (as also between (1) and (1′), and between (3) and (3′) is removed by a re-designation of the axes.

TABLE 9. RANGE OF r VALUES, AND CRITICAL FIELD CHARACTERISTICS, FOR GENERAL ELLIPSOID

Principal axes, a_1, a_2, a_3. Field in a_1, a_2 plane, decreasing from a positive value greater than $|H_0|$ and $|H_0^*|$. a_1 axis at an angle θ to the positive direction of the field, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. N_1, N_2, N_3, demagnetization coefficients along a_1, a_2, a_3. $r = (N_3 - N_2)/(N_1 - N_3)$. H_0^*, critical field at which I_0 leaves the a_1, a_2 plane. H_0, critical field when I_0 is restricted to the a_1, a_2 plane.

	$N_1 < N_2$	r range	characteristics
(1)	$N_1 < N_2 < N_3$	$-1 < r < 0$	I_0 in a_1, a_2 plane, all H
(2)	$N_1 < N_3 < N_2$	$0 < r$	H_0^* negative
(3)	$N_3 < N_1 < N_2$	$r < -1$	H_0^* positive
	$N_2 < N_1$		
(1′)	$N_2 < N_1 < N_3$	$r < -1$	I_0 in a_1, a_2 plane, all H
(2′)	$N_2 < N_3 < N_1$	$0 < r$	H_0^* negative
(3′)	$N_3 < N_2 < N_1$	$-1 < r < 0$	H_0^* positive

Summary and discussion

The number of essentially distinct cases, whose characteristic features will now be summarized, is three.

(i) H in plane containing the two longer axes ((1) and (1′) of table 9). The axis associated with the largest demagnetizing factor is normal to this plane. Over the whole range of H, I_0 remains in the plane defined by the longer axes, and the variation of magnetization with field is as calculated for the prolate spheroid.

(ii) H in plane containing the longest and shortest axes ((2) and (2′) of table 9). The axis associated with the intermediate demagnetizing factor is normal to this plane. The critical value, H_0^*, of H is negative. For $|H_0^*| > |H_0|$, the magnetization curves are the same as those for I_0 confined to the plane (i.e. as in figure 6), but for $|H_0^*| < |H_0|$, the hysteresis curves are correspondingly narrowed. For the limiting case of the oblate spheroid, H_0^* is zero, and the downward and upward discontinuities of the hysteresis curve coincide at $H = 0$.

(iii) H in plane containing the two shorter axes ((3) and (3′) of table 9). The axis associated with the smallest demagnetizing factor is normal to this plane. H_0^* is positive. For H decreasing, I_0 leaves the plane defined by the shorter axes at $+|H_0^*|$, passes through the longest axis direction at $H = 0$ (the resolved value of the magnetization in the plane then being zero), and returns to the plane at $H = -|H_0^*|$.

The results given for the prolate spheroid (§ 4) together with the equations in this section relating to the critical values of the parameters for which I_0 leaves the principal plane in which H lies, enable the magnetization curves (i.e. the sequence of associated values of $I_0 \cos\phi$ and H in the principal plane) to be determined with numerical precision in all the above cases, except for the range of H values in (iii) for which I_0 does not lie in the principal plane. It is, however, clear that here the process is reversible and that the associated part of the magnetization curve over the range between $+|H_0^*|$ and $-|H_0^*|$ passes through, and is symmetrical about, the origin, the detailed form of the curve depending on the orientation of the field relative to the shorter axes, and on the dimensional ratios.

Considering now an assembly of ellipsoids orientated at random, the behaviour will correspond to an appropriate mean of that for H in the three principal planes. The major

point which emerges from the above discussion is that it is only if the ellipsoids have the very special form of oblate spheroids (with the sphere as a limiting form) that hysteresis effects (as distinct from discontinuities in magnetization) will completely disappear. For ellipsoids with three unequal axes hysteresis effects will always occur. The requirement for pronounced hysteresis, and the higher coercivities, is that one of the principal axes shall be considerably longer than the other two, when the hysteresis curves will be similar to those shown in figures 6 and 7. The hysteresis and coercivity become smaller as the two longer axes become more nearly equal (though, as mentioned in § 4, a dimensional ratio as low as 1·1 may still lead to coercivities of several hundred oersteds).

Denoting the principal axes, in order of decreasing length, by α, β, γ, the general order of magnitude of the coercivity is given by $(N_\beta - N_\alpha) I_0$, though for particular orientations discontinuities may occur at fields as great as $(N_\gamma - N_\alpha) I_0$. Apart from the wide variety in detail in the behaviour of ellipsoids with three unequal axes, the main physically significant possibilities seem to be adequately covered by the quantitative results which have been given for the limiting forms of the prolate and the oblate spheroid.

6. Conditions for single-domain ellipsoidal particles

The full development of the magnetic characteristics of ellipsoidal particles which have been described is possible only if the magnetization is constant throughout the particle in both magnitude and direction. Such a state does not necessarily correspond to an energy minimum (total energy and free energy need not be distinguished in the present connexion), for the magnetic potential energy intrinsic to the particle, or the demagnetizing field energy, may be reduced by a non-uniform rearrangement of the magnetization directions. Any such rearrangement, however, is accompanied by an increase in the interchange interaction, or Weiss molecular field, energy, which is a minimum when all the effective electron spins are parallel. By consideration of these two factors, which are of predominant importance in the cases in which interest is centred, it is possible to obtain estimates of the critical size of ellipsoidal particles below which the state of uniform magnetization corresponds to the lowest attainable energy. The effect of other factors will be indicated later.

Domain structure

The problem is part of the wider problem of the formation and structure of domains in ferromagnetics, a detailed discussion of which is outside the scope of this paper. It is the less necessary in view of surveys of the domain problem by Brown (1940, 1945). Further, in a recent paper by Kittel (1946), touching closely on the present particular problem, full references are included to earlier work, the energy relations known to be relevant to domain boundary formation are indicated, estimates are made, for particles of various shapes, of critical sizes below which a single-domain structure is energetically favoured, and a useful summary is given of experimental evidence bearing on the problem. The critical diameter of a spherical particle of iron-like material is estimated as about $1{\cdot}5 \times 10^{-6}$ cm., and that of a cylindrical rod, of dimensional ratio 10, as some five times as great. As Kittel adopts an arbitrary value of 3 erg.cm.$^{-2}$ for the energy per unit area of a boundary, these estimates, in common with most of those which have been published, are admittedly very rough.

Moreover, as is, in effect, explicitly stated, the general method followed is not suitable for application to particles whose linear dimensions are comparable with the width of a Bloch wall.

Estimates of orders of magnitude are often extremely useful. Estimates of the critical diameter of particles as corresponding, say, to 50 atoms, are, however, of little value if 'order of magnitude' is broadly interpreted, and a factor 'of order unity' may mean a factor which might be as low as 0·1 and as high as 10. In the following discussion, therefore, an attempt is made to assess the precision of the estimates which are made. It will appear, however, that precision is limited more by uncertainties in fundamental aspects of the theory of ferromagnetism, namely in the theory of interchange interaction effects, than by the special difficulties of the particular problem.

Interchange interaction energy in transition zones

As fundamental to the problem, and as the source of the major uncertainties, an expression is first obtained, in a form suitable for making numerical estimates from experimental data, of the excess interchange interaction energy associated with a variation of the direction of magnetization. To avoid irrelevant complications, it will be assumed that the magnetization vector, I_0, has no component in the z direction, and that its direction in the xy plane is constant for z constant, making an angle ϕ with the x axis. It will further be supposed that the crystal structure is cubic, and that the x, y, and z directions coincide with three cubic axes. For a simple cubic lattice with one effective electron per atom, if the interchange interaction energy is negligible except between nearest neighbours, p per atom, and for them equal to $\mp J_0$ per atom pair for parallel or antiparallel spins, the energy per unit volume is given by

$$E_W = -\tfrac{1}{2}n_v \sum_p J_0 \cos\epsilon = -\tfrac{1}{2}n_v J_0 \left\{6 - a^2\left(\frac{d\phi}{dz}\right)^2\right\},$$

where n_v is the number of atoms per unit volume, ϵ the angle between the spin directions of neighbouring atoms, and a the lattice constant. If the spins are parallel, the energy is given by $-\frac{1}{2}N_W I_0^2$, where N_W is the Weiss molecular field coefficient, so that

$$\tfrac{1}{2}n_v J_0 = \tfrac{1}{2}N_W I_0^2/p = \tfrac{1}{2}\cdot\tfrac{1}{6}N_W I_0^2, \tag{6·1}$$

and

$$E_W = -\tfrac{1}{2}N_W I_0^2 + \tfrac{1}{2}\cdot\tfrac{1}{6}N_W I_0^2 a^2 (d\phi/dz)^2. \tag{6·2}$$

This expression is not new. It can be derived, in a more general way, from the treatment of Landau & Lifshitz (1935), and is given, for example, by Elmore (1938). For other lattice types, the factor $\frac{1}{6}$ is changed. If, however, in place of the lattice constant a, a parameter a_1 is used, where a_1^3 is the volume per atom, the numerical coefficients (for nearest neighbour interaction in each case) differ little. For the simple cube, body-centred cube, and face-centred cube, they are $\frac{1}{6}$ (= 0·167), $2^{\frac{2}{3}}/8$ (= 0·198), and $4^{\frac{2}{3}}/12$ (= 0·210) respectively, so that for all cubic lattices, with one electron per atom, and nearest neighbour interaction, the variable part of the energy may be rewritten, with good approximation,

$$E_W = \tfrac{1}{10}N_W I_0^2 a_1^2 (d\phi/dz)^2. \tag{6·3}$$

In general, the effective number, q, of electrons per atom differs from unity (e.g. for iron, $q = 2{\cdot}22$ and for nickel, $q = 0{\cdot}60_5$), and the treatment of the interaction problem presents difficulties which have not yet been resolved. For approximate purposes the procedure

which seems simplest, and as consistent with definite knowledge as any other, is to treat these electrons as being, on the average, uniformly distributed, and to replace the volume per atom, a_1^3, by the volume per electron, (a_1^3/q), giving as a more general expression than (6·3),

$$E_W = \tfrac{1}{10} N_W I_0^2 (a_1^3/q)^{\frac{2}{3}} (d\phi/dz)^2. \tag{6·4}$$

If the contribution to the interchange interaction energy from pairs of atoms other than nearest neighbours is not negligible (as it has been assumed to be in the derivation of (6·3)) the numerical coefficient in the energy expression in the form involving N_W would be increased, but not significantly unless J_0 fell off very slowly (or at first increased) for distances greater than those between nearest neighbours. It may be concluded that (6·4) gives a lower limit for the energy, which, however, is unlikely to be exceeded by a factor of more than 2.

The next requirement is to obtain an expression for N_W in terms of experimentally determinable quantities. In the Weiss molecular field treatment (as adapted to electron spins as carriers of the magnetic moment), and to the first approximation in more recent treatments, whether of the Heisenberg or collective electron types,

$$N_W = \frac{\rho}{A} \frac{1}{I_{00}^2} R\theta = \frac{a_1^3}{(q\mu_B)^2} k\theta, \tag{6·5}$$

where ρ is the density, A the atomic weight, I_{00} the saturation intensity at absolute zero (equal to $q\mu_B/a_1^3$), θ the Curie temperature, and μ_B the Bohr magneton. The value of N_W estimated from (6·5) is probably too low. For nickel, for example, a collective electron treatment indicates a value some two to three times as great. There are, however, so many uncertain factors that the only reasonable course seems to be to adopt (6·5) as giving a lower limit to N_W. The final expression for the energy,

$$E_W = \tfrac{1}{10} k\theta I_0^2 \frac{a_1^5}{(q\mu_B)^2 q^{\frac{2}{3}}} \left(\frac{d\phi}{dz}\right)^2, \tag{6·6}$$

enables an estimate to be made in a precise manner from the experimental data. The energy is almost certainly not lower than the estimate, but it may be some three times as high. More precise estimates can hardly be made without further development of the basic theory, and even then a detailed theoretical investigation would probably be necessary for each particular material. Writing

$$E_W = C_W (d\phi/dz)^2, \tag{6·7}$$

the values of C_W in erg.cm.$^{-1}$ for iron and nickel at room temperature are 3·6 and $5{\cdot}3 \times 10^{-7}$ respectively.

Bloch walls

When interchange interaction and magneto-crystalline anisotropy jointly contribute to the energy of a transition region, the effective width of a 180° boundary for minimum energy may be calculated by standard methods (Becker & Döring 1939, pp. 187–192), as indicated in § 1 (i). Using the expression (6·6) for the interchange energy, and (1·3) for the magneto-crystalline energy, the estimated effective widths for boundaries parallel to a (100) plane are about $8{\cdot}4 \times 10^{-6}$ cm. and $2{\cdot}9 \times 10^{-5}$ cm. for iron and nickel respectively, and the corresponding energies per unit area about 0·84 and 0·36 erg.cm.$^{-2}$. These widths, which may be

denoted by δ_c, are appreciably greater than the critical diameters estimated below for single domain ellipsoidal particles, and since the ratio of the magneto-crystalline to the interchange energy in a transition zone of width δ is equal to $(\delta/\delta_c)^2$, justification is provided for the initial neglect of the magneto-crystalline energy in the following treatment.

Demagnetizing field energy

The demagnetizing field energy, E_D per unit volume, is equal to $-\mathbf{H}\cdot\mathbf{I}$, where $\mathbf{H}$ is derived from the scalar potential, Ω, given by Poisson's expression,

$$\Omega = \int (\mathbf{I}/r)\cdot\mathbf{n}_1\, dS - \int (\operatorname{div}\mathbf{I}/r)\, dv. \tag{6·8}$$

(Cf. Elmore 1938.) If the magnetization is uniform, the second term is zero. From the first term may be derived the standard expression for the energy of an ellipsoid of revolution uniformly magnetized in the direction of the polar axis, namely

$$E_D = \tfrac{1}{2}N_a I_0^2 = 2\pi D_a I_0^2, \tag{6·9}$$

where N_a is the usual demagnetization coefficient, and $D_a = N_a/4\pi$. This energy is reduced by a rearrangement of the directions of magnetization of the type already described. Taking the centre of the ellipsoid as the co-ordinate origin, and the polar axis as the x direction, the contribution arising from the second term remains equal to zero if I_0 has no z component, and in direction is constant in the xy plane for constant z, varying only with z. The contribution arising from the first term approximates to zero if the angle ϕ made by I_0 with x varies from $\phi = 0$ at $z = -b$ to $\phi = 2\pi$ at $z = +b$, where b is the equatorial semi-axis of the ellipsoid. (A variation in angle from 0 to π would leave the ellipsoid with a resultant magnetization along an equatorial axis, and an energy which might be greater than in the initial state.) A distribution of magnetization of this kind, with an approximately zero demagnetizing field energy, could not be formed unless the energy given by (6·9) is greater than the additional interchange interaction energy associated with such a distribution, and calculable from (6·6). It follows that a state of uniform magnetization along a polar axis is a state of minimum energy for a particle in the form of a prolate ellipsoid if

$$2\pi D_a I_0^2 < C_W(2\pi/2b)^2; \tag{6·10}$$

or, alternatively, as a necessary condition for the division of the particle into separate domains by boundary formation,

$$b^2 \ll \frac{\pi}{20}\frac{k\theta}{D_a}\frac{a_1^5}{(q\mu_B)^2 q^{\frac{2}{3}}}. \tag{6·11}$$

Insertion of the numerical values for iron ($\theta = 1040°$ K, $a_1 = 2{\cdot}27\times10^{-8}$, $q = 2{\cdot}22$) and nickel ($\theta = 630°$ K, $a_1 = 2{\cdot}22\times10^{-8}$, $q = 0{\cdot}60_5$) gives the following results:

$$\text{Fe: } b \ll 4{\cdot}35\times10^{-7}D_a^{-\frac{1}{2}}, \quad \text{Ni: } b \ll 1{\cdot}80\times10^{-6}D_a^{-\frac{1}{2}}. \tag{6·12}$$

Critical dimensions for single-domain particles

As the dimensional ratio, m, of the prolate spheroid, varies from 1 (for the sphere) to 10, $D_a^{-\frac{1}{2}}$ varies from 1·73 to 7·02. The critical value of b, from (6·12), varies from $0{\cdot}75\times10^{-6}$ to $3{\cdot}06\times10^{-6}$ for iron, and from $3{\cdot}12\times10^{-6}$ to $1{\cdot}26\times10^{-5}$ for nickel. The corresponding number

of atoms in the ellipsoid varies from $1{\cdot}54 \times 10^5$ to $1{\cdot}06 \times 10^9$ for iron, and from $1{\cdot}17 \times 10^7$ to $8{\cdot}10 \times 10^{10}$ for nickel. Over the whole of this range of m values the critical diameters, $2b$, are less than the widths of the usual Bloch walls, estimated above, so that the method used of treating the 'boundary', or transition region as, in effect, filling the whole ellipsoid, is justified. Moreover, inclusion of the effect of the magneto-crystalline energy would increase the estimates of b by only about 30 % for $m = 10$; for the smaller values of m the effect is negligible. For prolate ellipsoidal particles of equatorial diameter appreciably greater than the effective width of a Bloch wall a method similar to that used by Kittel of treating the critical size problem would be more appropriate. For still larger 'particles' (e.g. ellipsoidal specimens of quasi-homogeneous monocrystalline or polycrystalline ferromagnetic materials) many other factors would have to be taken into consideration in any adequate treatment of domain structure. It is not surprising that the estimates obtained here for the lower limits of size of 'isolated' ferromagnetic particles (10^5 to 10^{10} atoms) below which the structure is almost certainly single-domain should be lower than the sizes of domains required to account for the bulk of the change of magnetization in the Barkhausen effect with ordinary materials (say 10^{10} to 10^{15} atoms); for the conditions are entirely different. The important point which is indicated by the argument is that particles may be formed of aggregates of atoms in which the number of atoms is adequate for the full development of ferromagnetic properties so far as intrinsic magnetization is concerned, but which are yet well below a critical size (depending on shape) for which the formation of domain boundaries becomes energetically possible.

7. Physical bearing of results

The principles which have been developed are believed to be relevant to the interpretation of the magnetic properties of ferromagnetic materials of which the composition is heterogeneous, and in which the more strongly ferromagnetic phase is partly or wholly in the form of magnetically anisotropic isolated particles of such size and shape that the magnetization is essentially uniform throughout any particle, but may change in direction under the influence of an applied field. The size must be below a critical value, depending on shape, such that the particle constitutes a single domain, in which boundary formation is precluded. 'Isolated' is to be understood merely in the sense that there is sufficient magnetic discontinuity to ensure that two or more particles do not form a unit in which common boundary formation and movement can occur; it is not necessarily incompatible with a relatively high concentration of the more ferromagnetic phase (as, for example, in powder magnets). 'Particle', again, should not be taken too literally as necessarily implying a sharply defined surface, but also as including the sub-microscopic atomic aggregates formed, for example, in the pre-precipitation stages of dispersion hardening in many alloys.

In the derivation of the magnetization curves for single-domain particles, the treatment has been developed deliberately in connexion almost exclusively with a magnetic anisotropy of the particles due to shape, to avoid confusing the mathematical argument by reference to a variety of physical applications. As indicated in § 1 (ii), however, similar characteristics may arise from magneto-crystalline and strain anisotropy. These will first be considered, and it will be seen that in a number of important and representative cases all the numerical results which have been obtained are immediately applicable with a suitably modified

conversion factor relating the field, H, with the non-dimensional parameter, h. Very brief reference is then made to the main type of ferromagnetic materials for which the theoretical results obtained appear to be of particular significance.

(i) *Types of magnetic anisotropy*

(*a*) *Magneto-crystalline anisotropy.* In the particular case of hexagonal cobalt, with an easy direction of magnetization along the hexagonal axis, the magneto-crystalline energy is given to a first approximation by

$$E_C = K \sin^2 \psi, \tag{7·1}$$

where ψ is the angle between the magnetization vector, I_0, and the hexagonal axis. It is readily seen that for a spherical single domain particle, with easy axis at an angle θ with the field, the equilibrium equation is expressible as

$$\tfrac{1}{2} \sin 2(\phi - \theta) + h \sin \phi = 0, \tag{7·2}$$

where

$$h = HI_0/2K, \tag{7·3}$$

and ϕ is the angle between I_0 and H. The equation (7·2) is identical with the equation (2·10) forming the basis of the treatment of the problem of the prolate spheroid, so that all the tables and curves can be taken as applying to the present case, with the reinterpretation of H. For cobalt, $I_0 \doteqdot 1{\cdot}4 \times 10^3$, and $K \doteqdot 4{\cdot}2 \times 10^6$, so $h = 1$ corresponds to $H = 6000$. This means (see table 6) that for an assembly of spherical single domain particles of cobalt, with the easy axes orientated at random, in a non-ferromagnetic matrix, the value of H_c, the coercivity would be about 2900.

The treatment given here does not, of course, cover cubic crystals for which the magneto-crystalline energy is given by an expression of the type (1·3). Although a full treatment of the magnetization curves for single-domain spherical particles in this case would be complicated, it is easy to show that the coercivity, H_c, is equal to $\alpha K/I_0$, with α depending on direction, but having a maximum value of 2; this gives for the maximum coercivity, arising in this way, values of about 400 for iron and 200 for nickel. The coercivity for random orientation of the easy axes would be considerably less. It may be noted that a treatment of rotational hysteresis of the type under consideration was given by Akulov in 1933 (see also Akulov 1939), but as it was thought of in relation to macroscopic single crystals, in which boundary movement processes limit the manifestation of the purely rotational characteristics, the whole treatment has usually been regarded as having little practical applicability (cf. Becker & Döring 1939, pp. 110, 217). It does, however, become immediately relevant to the behaviour of single-domain particles.

(*b*) *Strain anisotropy.* For a single-domain spherical particle, of negligible magneto-crystalline anisotropy, and isotropic in respect of the saturation magneto-striction coefficient, λ, subjected to a uniform tension, σ, the dependence of the magneto-strain energy on the angle, ψ, between I_0 and σ is given (cf. Becker & Döring 1939, pp. 146, 216), apart from constant terms, by

$$E_\sigma = \tfrac{3}{2} \lambda \sigma \sin^2 \psi. \tag{7·4}$$

The equilibrium equation again assumes the basic form (7·2), with

$$h = HI_0/3\lambda\sigma. \tag{7·5}$$

Thus, with the reinterpretation of h, the results obtained for shape anisotropy are immediately applicable to strain anisotropy. If λ and σ are of the same sign (as for nickel under compression, and, to a first approximation, iron under tension) the magnetic characteristics are similar to those of the single domain prolate spheroid; if λ and σ are of opposite sign, to those of the oblate spheroid. For an assembly of domain particles, with random orientation of the strain directions, the magnetic characteristics would be similar to those shown in figure 7. To obtain estimates of the values of the coercivity which may arise in this way, the value usually taken as a probable upper limit for σ may be adopted, namely 200 kg.mm.$^{-2}$ (127 tons/sq. in.). It is unnecessary to discuss here the complications arising from the variation of λ with direction in most ferromagnetic crystals, or the experimental uncertainties in the values. Adopting the values $\lambda = 1{\cdot}8 \times 10^{-5}$ for iron, and $\lambda = -3{\cdot}3 \times 10^{-5}$ for nickel, the values of H for $h = 1$ are obtained from (7·5) as approximately 600 and 4000 respectively. For hexagonal cobalt (λ varies with direction from approximately $-0{\cdot}4$ to $-2{\cdot}0 \times 10^{-5}$) the value of H for $h = 1$ would be of the same order as for iron. It is particularly to be noted that the particle itself is regarded as uniformly strained. The effect is distinct in character from the effect of stress variations in modifying the boundary movement process which was considered in § 1 (i), and the treatment does not involve such arbitrary assumptions, for, at least in principle, the state of strain of a particle could be related to its shape and the material constants (including the lattice spacings) of the particle itself and the matrix in which it is embedded (cf. Nabarro 1940).

(*c*) *Shape anisotropy.* The magnetic characteristics arising from departures of the shape of a particle from a spherical form have been sufficiently considered, and estimates of possible coercivity values, in particular, have been given towards the end of § 4. It is, however, desirable to indicate the degree of shape anisotropy which is required to produce effects comparable with those of magneto-crystalline and strain anisotropy which have just been estimated for spherical particles. For prolate spheroids of iron, the effect of shape anisotropy becomes more important than that of magneto-crystalline anisotropy for a dimensional ratio, m, greater than 1·05; and than that of magneto-strain anisotropy for $m > 1{\cdot}08$. For nickel, the crystalline effect is exceeded for $m > 1{\cdot}09$; but the maximum strain anisotropy effect is greater than that of shape anisotropy even for the highest dimensional ratios, this being due to the relatively high value of λ and the low value for I_0. For hexagonal cobalt, the magneto-crystalline effect is large, corresponding to a dipole rather than a quadrupole interaction, and for a comparable shape effect a dimensional ratio of about 3 would be required; in contrast, the strain effect is small, and comparable with that for iron.

In general, all three effects would be present to a greater or less extent, and even for those ferromagnetic materials whose magnetic properties appear to indicate, with some certainty, the presence of magnetically anisotropic single-domain particles, it may not be possible to determine unambiguously the major cause of the anisotropy without detailed consideration of fuller experimental evidence than is usually available. There are, however, two particular points which may be mentioned in connexion with the shape effect. The first is, that if the particles are iron-like, unless there is reason to suppose that the strain is extremely high, coercivities of more than about 400 provide strong presumptive evidence of a prolate spheroidal shape. The second is, that if there is reason to suppose that the particles are either plate or disk-like, or nearly spherical, so that the shape effect alone could not give rise

to appreciable hysteresis, coercivities of several hundred (or for certain particular materials even several thousand) would still be possible owing to magneto-crystalline and magneto-strain effects. Finally it must be borne in mind that in all the above estimates of the field, the field is that which is 'applied' to the particle, not that which is applied to the specimen. Owing to ordinary demagnetizing effects, and Lorentz field effects the two may differ enormously. Attention has been directed largely to coercivity, because the measured field at which the intensity is reduced to zero is, for obvious reasons, less subject to 'correction'. For an alloy containing more than one ferromagnetic phase, however, there is no simple or general relation between the measured coercivity of the alloy and that of the separate phases (cf. Gerlach 1938).

(ii) *Ferromagnetic materials*

A fairly thorough examination of the literature has shown that, while there are many ferromagnetic materials in which the magnetic anisotropy of single-domain particles may be contributory to the magnetic characteristics, the published papers do not usually contain sufficient information relevant to the present problem to enable irrefutable conclusions to be drawn. Many of the commercial alloys, in particular, are extremely complicated, and experimental work has largely been directed to the production of alloys with desired magnetic properties (by methods of trial which may be entirely appropriate for this purpose) rather than to the elucidation of how those properties arise. It would be out of place to give here a detailed discussion of even a small selection of these many materials. The theoretical considerations in this paper are to be regarded as giving a new line of approach to some of the problems presented by ferromagnetic materials, and not as providing a solution to all of them. A few notes on three of the types of materials to which the treatment seems to be particularly relevant may appropriately be given in concluding the paper.

(*a*) *Metals and alloys containing ferromagnetic impurities.* The effect of minute traces of ferromagnetic impurities has long been appreciated as a source of error in measurements of the magnetic properties of non-ferromagnetics, and various methods of correcting for the effect have been developed. More recently attention has been directed to the effect of various treatments (thermal, mechanical, heating in different atmospheres) on the state of the impurities themselves. This work has been reviewed by Constant (1945). In ordinary susceptibility determinations, magnetic measurements are usually made in very strong fields, and hysteresis effects would be unimportant, though reference has sometimes been made to the apparent magnetic 'hardness' of the impurities. The difficult experimental problem of measuring the coercivity of the impurities has been attacked by several investigators. For a specimen of brass (Cu, 45·5 Zn, $<0{\cdot}01$ Fe) Schröder (1939) found a coercivity of between 400 and 500, and for zinc with 1 % aluminium, a coercivity of 570. For brass, with a remanent intensity of only 0·08 (giving an indication of the iron content), Constant & Formwalt (1939) found a coercivity of 500, and for silver, copper and bismuth, all with remanent intensities less than 0·002, coercivities of 90, 200 and 300 respectively. With the very small amounts of iron in these alloys, these coercivities are most naturally explained in terms of the presence of single-domain particles; the smaller coercivities could arise from magneto-crystalline or strain anisotropy, but the occurrence of measured coercivities greater than 500 suggests that shape anisotropy is also involved.

(*b*) *Powder magnets.* An obvious application of the general ideas of this paper is to permanent magnets made of ferromagnetic powders or particles pressed together with a suitable binding medium. Magnets made of sintered powders of permanent magnet alloy material are finding increasing practical application (cf. Hoselitz 1946), but these are of less theoretical interest than magnets made up of small particles of, say, pure iron. If the particles are small enough to be of single-domain character, and are sufficiently distorted from a spherical to a prolate spheroidal form, such magnets, with a sufficiently large 'filling factor', offer a theoretical possibility of $(BH)_{\max.}$ values greatly exceeding those at present obtained. Unfortunately, very little information could be found about these magnets, though it was known that they had been greatly developed in France in recent years. It was not therefore surprising that, while the present section was being written, two short papers appeared by Néel (1947) dealing with precisely this problem; the first with the calculation of the critical size for single domain grains, and the second with the coercive field of a ferromagnetic powder with anisotropic grains. Néel's line of attack is much the same as that adopted here, and some of the results which he quotes as having been obtained by graphical calculation (e.g. an h_c value of 0·48 for randomly orientated prolate spheroids) are in close agreement with those found here. It is very satisfactory that entirely independent approaches should have led to similar theoretical conclusions. Since the information available to Néel on powder magnets was much more detailed than any which was previously accessible, it will be sufficient to refer to his papers (and to others which will doubtless follow) for a fuller treatment of this particular problem.

(*c*) *High coercivity alloys.* Among the many high coercivity alloys of the dispersion hardening type the one which has been most extensively investigated is the iron-nickel-aluminium alloy of approximate composition Fe_2NiAl, which is the prototype of a number of commercial permanent magnet alloys. The X-ray evidence obtained by Bradley & Taylor (1938*a*) has been interpreted by them (1938*b*; see also Bradley 1940) as indicating that high coercivity (values of from 500 to 600 are obtained) is associated with the initial stages of the breaking up of a single high temperature phase into two phases, β and β_2, the first approximating to pure iron, and the second to FeNiAl. A magnetic study of the system has been made by Sucksmith (1939), who represents the particular transition by

$$Fe_{95}Ni_{2\cdot5}Al_{2\cdot5}+2\cdot25(Fe_{30}Ni_{35}Al_{35})\rightleftarrows 3\cdot25(Fe_{50}Ni_{25}Al_{25}),$$

and gives the specific saturation intensities, σ_0, of the two phases on the left as approximately 212 and 61 respectively. In the intermediate high coercivity stage Bradley pictures the alloy as containing iron-rich 'islands', too small to be regarded as a precipitated phase in the usual sense, and retaining practically the same lattice spacing as in the original state, and so 'under a condition of immense strain'. This general picture is entirely compatible with an interpretation of the kind here suggested in terms of magnetically anisotropic single domain particles in a less ferromagnetic matrix; the type of strain required on the basis of this treatment (in contrast to the boundary movement treatment) is consistent with Bradley's tentative conclusions. The high coercivity suggests again that shape anisotropy as well as magneto-strain anisotropy is probably involved, but no definite conclusions can be drawn from the available evidence. It may perhaps be suggested, in connexion with the good permanent magnet qualities of this type of alloy, that the single-domain character of the

more ferromagnetic segregates is rendered possible by the excess of the less ferromagnetic phase, so that high coercivity can be developed; at the same time the main phase is sufficiently ferromagnetic to contribute significantly to the remanence of the alloy. It should be mentioned that Bradley's scheme of the constitution of the alloy is not accepted in detail by all other workers in the field (e.g. Snoek 1939), though the single domain particle interpretation of the magnetic properties seems equally compatible with the alternative scheme. Few conclusions seem to have been drawn from microphotographic studies of this alloy, though some of the photographs given by Kiuti (1941) appear to indicate the occurrence, under some conditions, of precipitate particles in the form of ellipsoids.

The preceding paragraph illustrates the fact that it is not possible to discuss adequately even one alloy with brevity; each alloy raises its own complex of problems. It is felt that it is better to leave the theoretical treatment of this paper to be taken into consideration in future studies of the magnetic properties of alloys, rather than to give here what would necessarily be an inadequate discussion of a large number of particular examples. It is, however, proper to state that there are a considerable number of alloys for which coercivities much higher than 600 have been reported, as in the systems Fe-Pt (1750), Co-Pt (3650), Fe-Nd (4300), Fe-Ag (5000), Ag-Al-Mn (5000); it seems very difficult to explain such high values along lines other than those suggested here.

There is, finally, one remarkable effect which should be mentioned in relation to the present treatment, namely the effect of cooling in a magnetic field on alloys of the alcomax or ticonal types. Demagnetization curves and some particulars of composition and treatment are given in a recent review article by Hoselitz (1946). The salient effect of cooling in a field is that in the direction in which the field has been applied the remanence, coercivity, and 'fullness' are all increased, and in a perpendicular direction decreased. Typical values for Alcomax II (Al 8, Ni 11, Co 24, Cu 6, rest Fe) are: along the field, B_r 12400, ${}_BH_c$ 570, $(BH)_{\max.}$ $4{\cdot}3 \times 10^6$; perpendicular, B_r 4600, ${}_BH_c$ 330, $(BH)_{\max.}$ $0{\cdot}43 \times 10^6$. The difference in the demagnetization curves in the two perpendicular directions is strikingly similar to the difference between the curves for single domain particles in the form of prolate spheroids with the easy directions approximately along the field (say the curve for $\theta = 10°$ in figure 6) and perpendicular to it ($\theta = 80°$). This suggests at once that cooling in a magnetic field favours a segregation of single domain particles or 'islands' with an easy direction of magnetization in the direction of the field. This is intelligible on energetic grounds, for the favoured orientation of the magnetically anisotropic segregate is that corresponding to the lowest energy. This would hold no less for magneto-crystalline or magneto-strain anisotropy than for shape anisotropy. There is an attractive simplicity in the idea that the segregates take the form of prolate spheroids with the long axes in the direction of the field. Far too little is known, however, about the segregation process, and the detailed constitution of the alloy after the magnetic treatment for a definite conclusion to be drawn at present about the type of magnetic anisotropy primarily involved. It must be sufficient, here as elsewhere, to have indicated directions which it may prove interesting to follow in future investigations.

References

Akulov, N. S. 1933 *Z. Phys.* **81**, 790.

Akulov, N. S. 1939 *Ferromagnetism* (in Russian). Moscow: State.

Becker, R. & Döring, W. 1939 *Ferromagnetismus.* Berlin: Springer. (Photo-lithoprint reproduction, 1943, Ann Arbor, Michigan: Edwards.)

Bradley, A. J. 1940 *Proc. Phys. Soc.* **52**, 80.

Bradley, A. J. & Taylor, A. 1938*a* *Proc. Roy. Soc.* A, **166**, 353.

Bradley, A. J. & Taylor, A. 1938*b* *Magnetism*, p. 89. Institute of Physics.

Brown, W. F. 1940 *J. Appl. Phys.* **11**, 160.

Brown, W. F. 1945 *Rev. Mod. Phys.* **17**, 15.

Comrie, L. J. (ed.) 1941 *Barlow's Tables.* London: Spon.

Constant, F. W. 1945 *Rev. Mod. Phys.* **17**, 81.

Constant, F. W. & Formwalt, J. M. 1939 *Phys. Rev.* **56**, 373.

Elmore, W. C. 1938 *Phys. Rev.* **53**, 757.

Gerlach, W. 1938 *Probleme der technischen Magnetisierungskurve* (ed. Becker, R.), p. 141. Berlin: Springer.

Hoselitz, K. 1946 *J. Sci. Instr.* **23**, 65.

Kersten, M. 1938 *Probleme der technischen Magnetisierungskurve* (ed. Becker, R.), p. 42. Berlin: Springer.

Kersten, M. 1943*a* *Phys. Z.* **44**, 63.

Kersten, M. 1943*b* *Grundlagen einer Theorie der ferromagnetischen Hysterese und der Koerzitivkraft.* Leipzig: Hirzel. (Photo-lithoprint reproduction, 1946, Ann Arbor, Michigan: Edwards.)

Kittel, C. 1946 *Phys. Rev.* **70**, 965.

Kiuti, S. 1941 *Sci. Rep. Tôhoku Univ.* **29**, 742.

Kondorsky, E. 1937 *Phys. Z. Sowjet.* **11**, 597.

Landau, L. & Lifschitz, E. 1935 *Phys. Z. Sowjet.* **8**, 153.

Nabarro, F. R. N. 1940 *Proc. Roy. Soc.* A, **175**, 519.

Neugebauer, P. V. (ed.) 1935 *Lohse: Tafeln für numerisches Rechnen mit Maschinen.* Leipzig: Engelmann.

Néel, L. 1947 *C.R. Acad. Sci., Paris*, **224**, 1488 and 1550.

Peters, J. 1942 *Seven-Place Values of Trigonometrical Functions.* New York: Van Nostrand.

Schröder, H. 1939 *Ann. Phys., Lpz.*, **36**, 71.

Snoek, J. L. 1939 *Physica*, **6**, 321.

Stoner, E. C. 1940 *Proc. Phys. Soc.* **52**, 175.

Stoner, E. C. 1944 *J. Instn Elect. Engrs*, **91**, 340.

Stoner, E. C. 1945 *Phil. Mag.* **36**, 803.

Sucksmith, W. 1939 *Proc. Roy. Soc.* A, **171**, 525