Entanglement and area laws in weakly correlated states*

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* Presentation based on refs [1, 2, 3, 4]

Here we will discuss the evaluation of entanglement measures in weakly correlated gaussian states. It will be shown how they can be expressed in terms of the singular values of a particular block of the generalized contraction matrix. This result enables to obtain in a simple way asymptotic expressions and related area laws for the entanglement entropy of bipartitions in pure states, as well as for the logarithmic negativity associated with bipartitions and also pairs of arbitrary subsystems. As an illustration, we consider different types of contiguous and noncontiguous blocks in two dimensional lattices. Exact asymptotic expressions are provided for first neighbor couplings, which lead to area laws depending on the orientation and separation of the blocks.

Gaussian States

- Belong to an infinite-dimensional Hilbert Space.
- Are the equilibrium states of harmonic systems.
- Closed under unitary evolution in harmonic systems.
- Arise as semi-classical approximations of the equilibrium states of quite general quamtum systems.
- Typical states in the context of Continuous Variable Quantum Information.
- Wick theorem: Completely determined by local expectation values and pair correlations between modes.

Formalism

Correlations in gaussian states are completely determinated by its *Generalized contraction matrix*:

$$\mathcal{D} = \langle \mathcal{Z}\mathcal{Z}^\dagger\rangle - \mathcal{M} = \begin{pmatrix} F^+ & F^- \\ \bar{F}^- & \bar{F}^+ + \mathbf{1}\end{pmatrix} \quad (1)$$

where $\mathcal{Z} = \left(\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{a}^\dagger, \ldots, \mathbf{a}_n^\dagger,\right)^t$
$\mathcal{M} = \mathcal{Z}\mathcal{Z}^\dagger - [(\mathcal{Z}^\dagger)^t\mathcal{Z}^t]^t = \left(\begin{smallmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & -\mathbf{1}\end{smallmatrix}\right)$ is the symplectic metric and
$F^+_{jk} = \langle \mathbf{a}_k^\dagger \mathbf{a}_j\rangle_\rho$ $\quad F^-_{jk} = \langle \mathbf{a}_k \mathbf{a}_j\rangle_\rho$. For a *pure gaussian state* $F^-\bar{F}^- = F^+ + (F^+)^2$

By means of a *Bogoliubov Canonical Transformation* it is possible to find a new basis for the bosonic álgebra such that $\mathcal{M}$ remains invariant and the correspondent F^- vanishes. In such a representation, the eigenvalues $\{f^\alpha\}$ of F^+, known as the *Symplectic Eigenvalues*, define a set of invariants associated to the state, related with its degree of mixness.

Entanglement in Gaussian States

For a *Gaussian Pure State*, the entanglement between a subsystem $\mathcal{A}$ and its complement $\bar{\mathcal{A}}$ is given by the entropy of any of both subsystems[5, 6]:

$$\mathcal{E}_{\mathcal{A}\bar{\mathcal{A}}} = S_{\mathcal{A}} = S_{\bar{\mathcal{A}}} = \sum_\alpha h(f^\alpha_{\mathcal{A}}) \quad (2)$$

where $f^\alpha_{\mathcal{A}}$ are the symplectic eigenvalues associated to $\mathcal{D}_{\mathcal{A}}$ (the contraction matrix of the subsystem) and $h(x) = -x\log x + (1+x)\log(x+1)$ is a convex function.

For non pure states or non complementary subsystems $\mathcal{B}$, $\mathcal{C}$, a measure of entanglement is given by the Logarithmic Negativity [7, 8]

$$\mathcal{E}^{\mathcal{N}}_{\mathcal{BC}} = \log\|\rho^{t_{\mathcal{B}}}_{\mathcal{BC}}\|_1 = \sum_{\alpha/\tilde{f}^\alpha_{\mathcal{BC}}<0} \log(1 + 2\tilde{f}^\alpha_{\mathcal{BC}}) \quad (3)$$

where $\tilde{f}^\alpha_{\mathcal{BC}}$ are the *negative* symplectic eigenvalues of the contraction matrix $\tilde{\mathcal{D}}_{\mathcal{BC}}$ associated to the density matrix $\rho^{t_{\mathcal{B}}}_{\mathcal{BC}}$.

As the partial transposition is equivalent in this context to change $\mathbf{a}_k \leftrightarrow \mathbf{a}_k^\dagger$ for each k in the subsystem $\mathcal{B}$ and revert its order in each product, $\tilde{\mathcal{D}}_{\mathcal{BC}}$ has blocks $\tilde{F}^\pm_{\mathcal{BC}}$ given by

$$\tilde{F}^\pm_{\mathcal{BC}} = \begin{pmatrix} \bar{F}^\pm_{\mathcal{B}} & \bar{F}^\mp_{\mathcal{B,C}} \\ F^\mp_{\mathcal{C,B}} & F^\pm_{\mathcal{C}}\end{pmatrix} \quad (4)$$

Weakly correlated Gaussian States

At the lowest order (in the strength of the pair-correlations $F^\pm$), the entropy of a subsystem for a global pure Gaussian State is a function of the singular values $\{\sigma_\alpha\}/\det((F^-_{\mathcal{A},\bar{\mathcal{A}}})^\dagger F^-_{\mathcal{A},\bar{\mathcal{A}}} - \sigma_\alpha^2\mathbf{1}_{\bar{\mathcal{A}}}) = 0$ of the submatrix

$$(F^-_{\mathcal{A},\bar{\mathcal{A}}})_{ij} = \langle a_j a_i\rangle$$

(i.e the submatrix of F^-_{ij} with the i (j)-index associated to the subsystem $\mathcal{A}$ ($\bar{\mathcal{A}}$))[1].

$$\mathcal{E}_{\mathcal{A},\bar{\mathcal{A}}} \approx \sum_\alpha h\left((\sigma^\alpha_{\mathcal{A},\bar{\mathcal{A}}})^2\right) \quad (5)$$

and the log-negativity of a (non-complementary) partition $\mathcal{BC}$ is given by [1]

$$\mathcal{E}^{\mathcal{N}}_{\mathcal{BC}} \approx -2\log_2(e)\sum_{\tilde{f}^{\mathcal{BC}}_\alpha<0} \tilde{f}^{\mathcal{BC}}_\alpha \quad (6)$$

where

$$\tilde{f}^\alpha_{\mathcal{B,C}} \approx \max\left(0, -\sigma^{\mathcal{B,C}}_\alpha + \frac{(\bar{G}_{\mathcal{B}})_{\alpha\alpha} + (G_{\mathcal{C}})_{\alpha\alpha}}{2}\right)$$

being

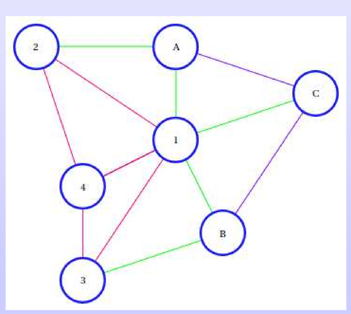
$$G_{\mathcal{S}} = \tilde{F}^+_{\mathcal{S}} - \tilde{F}^-_{\mathcal{S}}\bar{\tilde{F}}^-_{\mathcal{S}} \approx \tilde{F}^-_{\mathcal{S},\bar{\mathcal{S}}}\bar{\tilde{F}}^-_{\bar{\mathcal{S}},\mathcal{S}}$$

i.e. $G_{\mathcal{S}}$ is taking into account the effect of the environment over the effective modes entangled between $\mathcal{B}$ and $\mathcal{C}$.

Area laws

Suppose now that correlations are "local", so we can think of F^- as a *adjacency matrix* $F^-_{ij} \approx f_0 K_{ij}$

$$K = \begin{pmatrix} - & 1 & 2 & 3 & 4 & A & B & C \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ A & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ B & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ C & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$



where f_0 is a constant and K_{ij} takes values 1(0) depending on the modes i, j are (not) adjacent.
Calling $K_{\mathcal{AB}}$ the green subblock, the number of shared links between $\mathcal{A}$ and $\mathcal{B}$ is given by

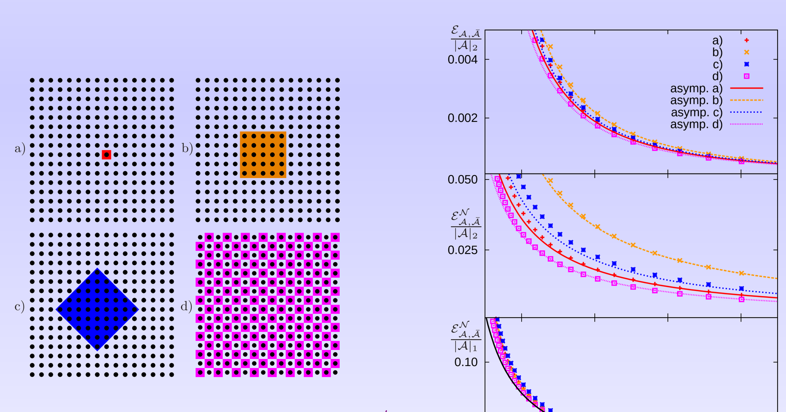
$$n_{\mathcal{AB}} = \mathrm{Tr}\left[K^\dagger_{\mathcal{AB}}K_{\mathcal{AB}}\right] = \sum_\alpha \sigma_\alpha^2 = \|K_{\mathcal{AB}}\|_2^2 \quad (7)$$

Assuming the number of shared pairs is proportional to the *minimum area* of a surface separating both subsystems, the area law [9, 10] is satisfied trivially for this quantity.

Bipartitions

$$\mathbf{H}_b = \sum_i \lambda\mathbf{b}_i^\dagger\mathbf{b}_i - \sum_{\{i,j\}}\left(\mathbf{b}_i^\dagger\mathbf{b}_j + \frac{1}{3}\mathbf{b}_i^\dagger\mathbf{b}_j^\dagger + h.c.\right) \quad (8)$$

Scaling of the entanglement entropy and Log Negativity for this lattice



$$|\mathcal{A}|_\gamma = \mathrm{Tr}[(K^t_{\mathcal{A}\bar{\mathcal{A}}}K_{\mathcal{A}\bar{\mathcal{A}}})^{\gamma/2}]$$

FIGURE 1: (From [1]) Scaling laws for $S_{\mathcal{A}}$ and $\mathcal{E}^{\mathcal{N}}_{\mathcal{A}}$ for different partitions.

Non complementary partitions

Scaling of the entanglement entropy and Log Negativity for this lattice

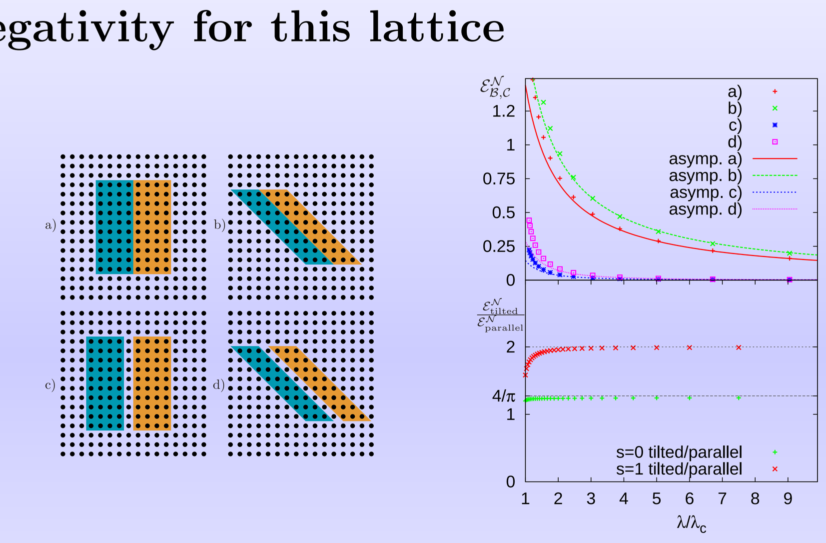


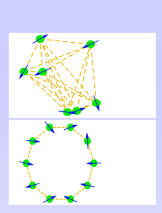
FIGURE 2: (From [1]) Log negativity for different non-complementary partitions. Due to the effect of the environment, the leading order of $\mathcal{E}^{\mathcal{N}}_{\mathcal{BC}}$ is higher than the contiguous case. Also we can notice that the quotient of $\mathcal{E}^{\mathcal{N}}_{\mathcal{BC}}$ for subsystems sharing tilted and parallel boundaries do not coincides with the correspondent quotient of "euclidean" areas.

Extension to spin systems through the RPA bosonization method

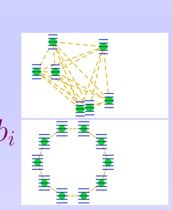
Through the Random Phase Approximation + Symmetry Restoration method, it is possible to connect the previous results to more general cases, as spin systems.
As an example, we will consider a spin-**s** system with actractive first neighbour interactions in a transverse magnetic field[4, 2]:

$$\mathbf{H}_s = \sum_i b\mathbf{s}_{zi} - \sum_{\mu=x,y}\sum_{\{i,j\}}\frac{v_\mu}{\sqrt{2s}}\mathbf{s}_{\mu i}\mathbf{s}_{\mu j} \quad (9)$$

For spin systems at zero temperature, the RPA bosonization is consistent with the approximate local bosonization



$$\begin{aligned} S_{+i} &\to \sqrt{s_i}b_i^\dagger \\ S_{-i} &\to \sqrt{s_i}b_i \\ S_{zi} &\to -s_i + b_i^\dagger b_i \\ |MF\rangle &\to |0\rangle_b \end{aligned}$$



At second order in the bosonic operators, the ground state of the bosonized Hamiltonian is *Gaussian*, and because the bosonization is - at this order - a local unitary transformation, each entanglement measures are preserved:

$$|\mathrm{GS}\rangle_b = \mathcal{N}_b\exp(Z_{ij}\mathbf{b}_i^\dagger\mathbf{b}_j^\dagger)|0\rangle \leftrightarrow |\mathrm{RPA}\rangle = \mathcal{N}_{RPA}\exp\left(\frac{Z_{ij}}{\sqrt{s_is_j}}\mathbf{s}_i^\dagger\mathbf{s}_j^\dagger\right)|MF\rangle \quad (10)$$

If the mean field problem is degenerated due to an expontaneus symmetry breaking, a more accurated ground state estimation is given by

$$|\mathrm{SRRPA}\rangle = \sum\!\!\!\!\!\!\int \mathbf{R}_g|\mathrm{RPA}\rangle dg \quad (11)$$

where the integration is over the symmetry group, R_g is certain representation of the symmetry group and $|\mathrm{RPA}\rangle$ is the RPA state built over one of the mean field solutions.

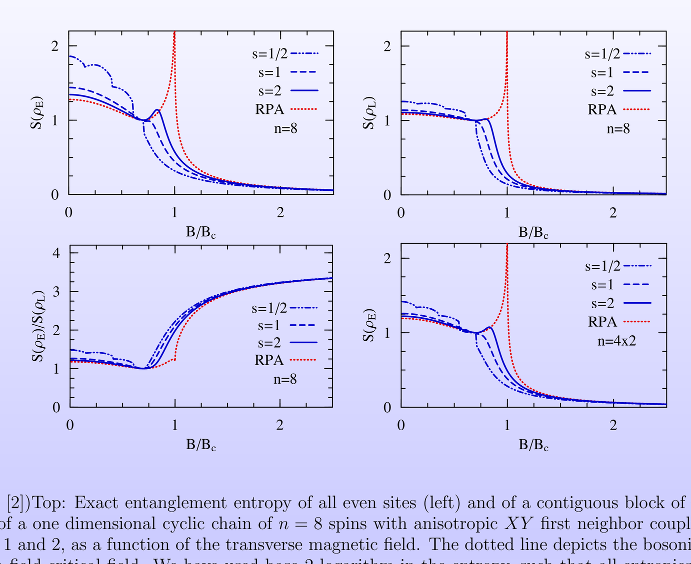


FIGURE 3: (From [2])Top: Exact entanglement entropy of all even sites (left) and of a contiguous block of $n/2$ sites (right) in the ground state of a one dimensional cyclic chain of $n = 8$ spins with anisotropic XY first neighbor couplings ($J_y/J_x = 1/2$) and spin $s = 1/2$, 1 and 2, as a function of the transverse magnetic field. The dotted line depicts the bosonic RPA result, with $B_c = J_x$ the mean field critical field. We have used base 2 logarithm in the entropy, such that all entropies approach 1 at the factorizing field $B_s \approx 0.71B_c$. Bottom: Left: The corresponding ratio $S(\rho_E)/S(\rho_L)$. Right: The entanglement entropy of all even sites in a rectangular lattice of 4×2 spins. Remaining details as in the top panels.

For low field, when the mean field problem is degenerated, it can be shown that the local entropy is well approximated as $S_{\mathcal{A}} = S^{(b)}_{\mathcal{A}} + S_{MF}$ where $S^{(b)}_{\mathcal{A}}$ is the local entropy estimated by the bosonization and S_{MF} is an almost constant term coming from the degeneration of the mean field.

Generalizations

- The case of global non pure weakly correlated gaussian states can be recovered by purification of the global state.
- The implementation for other measures of quantum correlatios like mutual information or quantum discord, as well as for continuous systems are currently in progress.

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