Generalized conditional entropy in bipartite quantum systems

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Abstract

We analyze the conditional entropy of a bipartite quantum system A+B for a general concave entropic form, obtained as a result of a local measurement on one of the systems (B). We derive some universal properties of this generalized conditional entropy and show that its minimum over all local measurements represents the asociated entanglement of formation between A and a purifying third system C. We also consider the case of linear entropy, where a closed expression for the minimum among projective measurements in the case of a qudit-qubit state is derived.

Bipartite system A+B

Average uncertainty about A, after a local measurement on system B defined by a set of operators $\Pi_j = I_A \otimes \Pi_j^B$

$$\Pi_j^B = r_j |j_B\rangle\langle j_B|, \ \sum_j \Pi_j = I$$

Conditional *f*-entropy [5]

$$S_f(A|B_{\{\Pi_j\}}) = \sum_j p_j S_f(\rho_{A/\Pi_j})$$

Where $p_j = Tr\rho\Pi_j$ $\longrightarrow$ $p_j = \frac{r_j}{d_B}(1+\boldsymbol{r}_B\cdot\boldsymbol{k}_j)$

$\rho_{A/\Pi_j} = p_j^{-1} Tr_B \rho \Pi_j$ $\qquad$ $\rho_{A/\Pi_j} = \frac{1}{d_A}\left[I_A + \frac{(\boldsymbol{r}_A + J\boldsymbol{k}_j)\cdot\boldsymbol{\sigma}_A}{1+\boldsymbol{r}_B\cdot\boldsymbol{k}_j}\right]$

Average *information gain* about A after a measurement in B

***f*-information gain [5]**

$$I_f(A|B_{\{\Pi_j\}}) = S_f(A) - S_f(A|B_{\{\Pi_j\}})$$

For any strictly concave function *f* satisfying *f(0)=f(1)=0*, the trace entropy of a quantum state ρ is given by [1]

$$S_f(\rho) = Trf(\rho)$$

$$\rho = \frac{1}{d}[I + \boldsymbol{r}_A\cdot\boldsymbol{\sigma}_A \otimes I_B + I_A \otimes \boldsymbol{r}_B\cdot\boldsymbol{\sigma}_B + \boldsymbol{\sigma}_A^t J \otimes \boldsymbol{\sigma}_B]$$

$$\boldsymbol{r}_A = \langle\boldsymbol{\sigma}_A\rangle, \boldsymbol{r}_B = \langle\boldsymbol{\sigma}_B\rangle, J = \langle\boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B^t\rangle$$

Measurement operators can be written as

$$\Pi_j^B = \frac{r_j}{d_B}(I_B + \boldsymbol{k}_j\cdot\boldsymbol{\sigma}_B)$$

$$\boldsymbol{k}_j = Tr_B(\boldsymbol{\sigma}_B |j_B\rangle\langle j_B|), \ \sum_j r_j \boldsymbol{k}_j = 0$$

$$S_2(\rho) = 1 - Tr(\rho^2)$$

In a system with Hilbert space dimension d, considering a set of hermitian operators $(I,\boldsymbol{\sigma})$ with $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{d^2-1})$, $\mathrm{Tr}\sigma_i = 0$, $\mathrm{Tr}\sigma_i\sigma_j = d\,\delta_{ij}$

A general state can be written as

$$\rho = \frac{1}{d}(I + \boldsymbol{r}\cdot\boldsymbol{\sigma}), \ \boldsymbol{r} = Tr\rho\boldsymbol{\sigma} = \langle\boldsymbol{\sigma}\rangle$$

And quadratic entropy is

$$S_2(\rho) = 1 - \frac{1}{d} - \frac{|\boldsymbol{r}|^2}{d}$$

Quadratic conditional entropy

$$S_2(A|B_{\{\Pi_j\}}) = S_2(A) - \frac{1}{d_A}\sum_j r_j \frac{\boldsymbol{k}_j^t C^t C \boldsymbol{k}_j}{1+\boldsymbol{r}_B\cdot\boldsymbol{k}_j}$$

Where C is the *correlation matrix*

$$C = J - \boldsymbol{r}_A \boldsymbol{r}_B^t = \langle\boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B^t\rangle - \langle\boldsymbol{\sigma}_A\rangle\langle\boldsymbol{\sigma}_B^t\rangle$$

Quadratic information gain [5]

$$I_2(A|B_{\{\Pi_j\}}) = \frac{1}{d_A}\sum_j r_j \frac{\boldsymbol{k}_j^t C^t C \boldsymbol{k}_j}{1+\boldsymbol{r}_B\cdot\boldsymbol{k}_j}$$

$$S_2(A|B_{\boldsymbol{k}}) = S_2(A) - \frac{1}{d_A}\frac{\boldsymbol{k}^t C^t C \boldsymbol{k}}{1-(\boldsymbol{r}_B\cdot\boldsymbol{k})^2}$$

$$I_2(A|B_{\boldsymbol{k}}) = \frac{1}{d_A}\frac{\boldsymbol{k}^t C^t C \boldsymbol{k}}{1-(\boldsymbol{r}_B\cdot\boldsymbol{k})^2}$$

Relation with the squared Hilbert-Schmidt distance

$$\|\rho - I/d\|^2 = |\boldsymbol{r}|^2/d = S_2(I/d) - S_2(\rho)$$

$$\|\rho - \rho_A \otimes \rho_B\|^2 = \frac{1}{d}\|C\|^2 = \frac{1}{d}TrC^tC$$

$$\sum_j p_j \|\rho_A - \rho_{A/\Pi_j}\|^2 = I_2(A|B_{\{\Pi_j\}})$$

Universal properties

$$S_f(A) \geq S_f(A|B_{\{\Pi_j\}})$$

If $\rho = \sum_\alpha q_\alpha \rho^\alpha$ and $S_f(A^\alpha|B^\alpha_{\{\Pi_j\}}) = \sum_j p_j^\alpha S_f(\rho^\alpha_{A/\Pi_j})$

$$S(A|B_{\{\Pi_j\}}) \geq \sum_\alpha q_\alpha S_f(A^\alpha|B^\alpha_{\{\Pi_j\}})$$

$$S_f(A|B_{\{\Pi_j\}}) \geq S_f(A|B_{\{\tilde{\Pi}_k\}})$$

if $\Pi_j = \sum_k r_j^k \tilde{\Pi}_k, \sum_j r_j^k = 1$

For states of the form $\rho = w|\Psi\rangle\langle\Psi| + (1-w)I_d/d$ the minimum conditional entropy is obtained for a measurement in the pointer basis $\{|\tilde{k}_B\rangle\}$ formed by the eigenstates of $\rho_B = Tr_A(\rho)$

$$S_f(A|B) = S_f(A|B_{\{\tilde{\Pi}_k\}})$$

Minimum conditional entropy

$$S_f(A|B) \equiv \underset{\{\Pi_j\}}{Min}\, S_f(A|B_{\{\Pi_j\}})$$

$$I_f(A|B) = S_f(A) - S_f(A|B)$$

Defining $N_B = I_3 - \boldsymbol{r}_B \boldsymbol{r}_B^t$ and $\tilde{C} = CN_B^{-1/2}$

$$\frac{\boldsymbol{k}^t C^t C \boldsymbol{k}}{\boldsymbol{k} N_B \boldsymbol{k}} \leq \tilde{\boldsymbol{k}}\, \tilde{C}^t \tilde{C}\, \tilde{\boldsymbol{k}} = \lambda_{max}$$

Where λ_{max} is the maximum eigenvalue of $\tilde{C}^t\tilde{C}$ and $\tilde{\boldsymbol{k}}$ the asociated normalized eigenvector.

We have then

$$S_2(A|B) = S_2(A) - \frac{1}{d_A}\lambda_{\max}$$

$$I_2(A|B) = \frac{1}{d_A}\lambda_{\max}$$

Minimum quadratic conditional entropy and information gain for a A+B qudit-qubit system, as a result of a projective measurement on B

Minimum conditional entropy and Generalized entanglement of formation

If system A+B is purified by adding a third system C, the minimum conditional entropy equals the entanglement of formation between A and C

$$S_f(A|B) = E_f(A,C) = \underset{\sum_j p_j \rho^j_{AC} = \rho_{AC}}{Min} \sum_j p_j S_f(\rho_j^A)$$

Application to X states [4]

For states of the form $\rho = \begin{pmatrix} p_+ & 0 & 0 & \alpha_- \\ 0 & q_+ & \alpha_+ & 0 \\ 0 & \alpha_+ & q_- & 0 \\ \alpha_- & 0 & 0 & p_- \end{pmatrix}$ $\quad p_\pm = \frac{1\pm(r_A+r_B)+J_z}{4}$, $q_\pm = \frac{1\pm(r_A-r_B)-J_z}{4}$, $\alpha_\pm = \frac{J_x \pm J_y}{4}$

Matrices C and N_B are $C = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z - r_A r_B \end{pmatrix}$, $N_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r_B^2 \end{pmatrix}$

Wich leads to $S_2(A|B) = 1 - |\boldsymbol{r}_A|^2 - I_2(A|B)$

$$I_2(A|B) = \underset{\boldsymbol{k}}{Max}\frac{\boldsymbol{k}^t C^t C \boldsymbol{k}}{\boldsymbol{k}^t N_B \boldsymbol{k}} = \mathrm{Max}\left[J_x^2, J_y^2, \frac{(J_z - r_A r_B)^2}{1 - r_B^2}\right]$$

As a particular case we may consider the state

$$\rho = \frac{1}{2}(|\theta\theta\rangle\langle\theta\theta| + |-\theta\theta\rangle\langle-\theta-\theta'|)$$

Where $|\theta\rangle = \exp[-i\theta\sigma_y/2]|0\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$

This is an X state with

$$r_A = r_B = \cos\theta, J_z = \cos^2\theta, J_x = \sin^2\theta, J_y = 0$$

We then obtain that $S_2(A|B_k)$ is minimized for $\boldsymbol{k}$ *along x* $\forall\,\theta$, leading to

$$S_2(A|B) = 1 - cos^2\theta - sin^2\theta = \frac{1}{4}sin^2 2\theta$$

$$I_2(A|B) = sin^4\theta$$

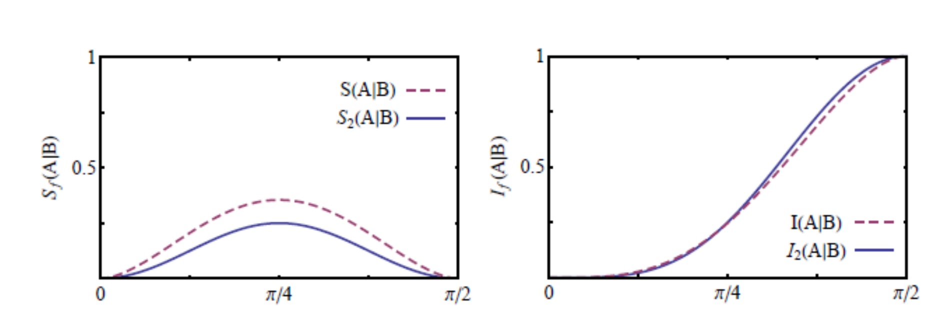

Fig. 2. Results for the quadratic (solid lines) and von Neumann (dashed lines) minimum conditional entropy (left) and maximum information gain (bottom) in the mixture of aligned states.

Writing $|\Psi\rangle = \sqrt{q}|00\rangle + \sqrt{1-q}|11\rangle$

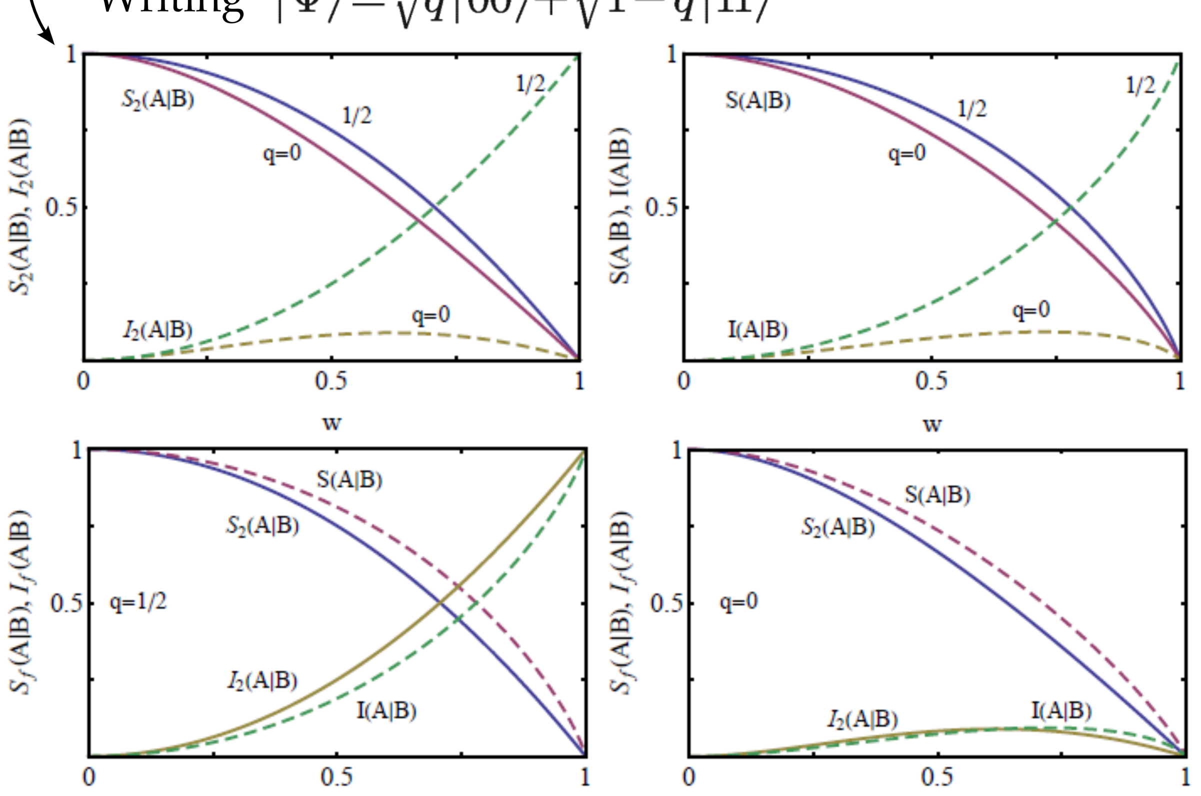

Fig. 1. Top: results for the quadratic (left) and the von Neumann (right) minimum conditional entropy (solid lines) and maximum information gain (dashed lines) after a measurement at B in the mixture of a pure state with the maximally mixed state. All $S_f(A|B)$ are concave decreasing functions of w, vanishing in the pure state limit w=1. Bottom: comparison between the quadratic (solid lines) and von Neumann based (dashed lines) results for q=1/2 (left) and q=0 (right). It is verified that $S_2(A|B) \leq S(A|B)\ \forall\, w,q$.

Conclusions

We have analyzed the main features of the conditional entropy associated to general concave entropic forms in bipartite quantum systems, as determined by a measurement in one of the constituents. Its minimum among all local measurements represents the entanglement of formation $E_f(A,C)$ between A and a purifying third system C. For some classes of states the minimizing measurement is unique for all S_f.

A main result of our work is the closed evaluation of this minimum for the case of the linear entropy in a general *A+qubit* system. In the examples considered, the S_2 optimizing measurement is the same as that minimizing von Neumann entropy and results for both entropies are completly similar.

Legend

$\dashrightarrow$ $f(\rho) = \rho - \rho^2$

$\dashrightarrow$ Concavity

$\dashrightarrow$ Projective measurement

$\longrightarrow$ consider a set of operators $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{d^2-1})$ with $Tr\sigma_i = 0$, $Tr\sigma_i\sigma_j = d\,\delta_{ij}$

References

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