Quantum Discord and Information Deficit in finite XX chains

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We examine the quantum correlations of spin pairs in the exact ground state of cyclic finite XX spin chains in a transverse field, by evaluating the Quantum Discord [1,2], the Geometric Discord [3] as well as the Information Deficit [4]. We show that parity effects are of crucial importance for describing the behavior of these measures below the critical field, and imply full range in the immediate vicinity of the factorizing field, where they become independent of separation and coupling range.
It is also shown that while these measures exhibit the same qualitative asymptotic behaviour for large separations or temperatures, at the same time important differences arise in the minimizing local measure that defines them. Whereas the Quantum Discord prefers a spin measurement perpendicular to the transverse field, the Geometric Discord and the Information Deficit exhibit a perpendicular to parallel transition as the field increases below the critical field, which subsists at all temperatures and for all separations. We show exact results of these measures for XX and XY spin chains, both for finite chains and for the thermodynamic limit obtained by means of the Jordan Wigner femionization.

General formalism

Let us consider a bipartite quantum system $A+B$ in a state ρ_{AB}. A local complete projective measurement on system B, M_B is defined by a set of orthogonal projectors $\{\Pi^B_j\}$, $\Pi^B_j = I_A\otimes\Pi_j$, with $\Pi^B_j\Pi^B_{j'} = \delta_{jj'}\Pi^B_j$ and $\sum_j \Pi^B_j = I$.

The state of the total system after an unread measurement is

$$\rho'_{AB} = \sum_j \Pi^B_j\,\rho_{AB}\,\Pi^B_j = \sum_j q_j\,\rho_{A/j}\otimes\Pi_j \qquad (1)$$

The minimum generalized information loss due to such measurement is [5]

$$I^B_f = \underset{M_B}{\text{Min}}\, S_f(\rho'_{AB}) - S_f(\rho_{AB}) \qquad (2)$$

where $S_f(\rho)$ is a general entropic form [5]:

$$S_f(\rho) = \text{Tr}\, f(\rho) \qquad (3)$$

with f smooth concave function in $[0,1]$ and $f(0)=f(1)=0$. It is verified that $I^B_f \geq 0$, for any f with $I^B_f = 0$ if $\rho'_{AB} = \rho_{AB}$.
Positivity of I^B_f $\forall\, S_f$ follows from majorization relation $\rho'_{AB} \prec \rho_{AB}$ satisfied by (1).

Particular entropic functions

For $f(\rho) = -\rho\log_2\rho$, $S_f(\rho)$ is the von Neumann entropy $S(\rho)$ and (2) becomes the one way information deficit (I^B_1) [4].
It can be rewritten in terms of the relative entropy $S(\rho||\rho') = -\text{Tr}\,\rho(\log_2\rho' - \log_2\rho)$ as

$$I^B_1 = \underset{M_B}{\text{Min}}\, S(\rho'_{AB}) - S(\rho_{AB}) = \underset{M_B}{\text{Min}}\, S(\rho_{AB}||\rho'_{AB})\,. \qquad (4)$$

For pure states $\rho^2_{AB} = \rho_{AB}$, $I^B_1 = E(A,B)$ (entanglement entropy).

For $f(\rho) = \rho(1-\rho)$, $S_f(\rho)$ is the linear entropy and (2) is proportional to the Geometric Discord [3]

$$I^B_2 \;=\; \underset{\rho'_{AB}}{\text{Min}}\, ||\rho_{AB} - \rho'_{AB}||^2$$

For pure states $I^B_2 = C^2_{AB}$, with C_{AB} the concurrence

Relation with the Quantum Discord [1,2]:

$$D = \underset{M_B}{\text{Min}}\, I^{M_B}_1(\rho_{AB}) - I^{M_B}_1(\rho_B)$$

Implies $D^B \leq I^B_1$, with $D^B = I^B_1$ if $\rho'_B = \rho_B$ for minimizing M_B.

Determination of the minimizing measurement

Determination of M_B difficult in general. Complete projective measurements at B determined by $d^2_B - d_B$ real parameters if B has dimension d_B. Minimizing measurement for I^B_f fulfills stationary condition [5]

$$\text{Tr}_A[f'(\rho'_{AB}), \rho_{AB}] = 0\,, \qquad (5)$$

leading to $d_B(d_B-1)$ real equations.
Geometric discord I_2: For a general mixed state of two qubits

$$\rho_{AB} = \tfrac{1}{4}(I + \boldsymbol{r}_A\cdot\boldsymbol{\sigma}_A + \boldsymbol{r}_B\cdot\boldsymbol{\sigma}_B + \boldsymbol{\sigma}^t_A J\boldsymbol{\sigma}_B)\,, \qquad (6)$$

where $\boldsymbol{\sigma} = 2\boldsymbol{s}$ are the Pauli matrices, $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}\otimes I$, $\boldsymbol{\sigma}_B = I\otimes\boldsymbol{\sigma}$, $\langle\boldsymbol{\sigma}_{A,B}\rangle = \boldsymbol{r}_{A,B}$ and $J = \langle\boldsymbol{\sigma}_A\boldsymbol{\sigma}^t_B\rangle$, it can be shown that [5],

$$I^B_2 = \tfrac{1}{2}(\text{tr}\, M_2 - \lambda_1)\,, \qquad (7)$$

where λ_1 is the largest eigenvalue of the positive semi-definite 3×3 matrix $M_2 = \boldsymbol{r}_B\boldsymbol{r}^t_B + J^t J$. The minimizing M_B is a spin measurement along the direction of the associated eigenvector $\boldsymbol{k}_1$ of M_2. A closed expression for I^B_3 can also be obtained for this case [5].

XY spin chain in a transverse field I

Finite spin $1/2$ array with XY couplings in a transverse field:

$$H = \sum_i B^i s_{i,z} - \tfrac{1}{2}\sum_{i,j} J^{ij}_x s_{ix}s_{jx} + J^{ij}_y s_{iy}s_{jy}, \qquad (8)$$

$[H,P_z]=0$, $P_z = \exp[i\pi S_z]$. Pair reduced state in GS then satisfies [6] $[\rho_{ij}, P^{ij}_z] = 0\,,\;\; P^{ij}_z = \exp[i\pi(s_{iz}+s_{jz}+2s)]$, and implies definite parity reduced eigenstates. Hence, in standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$,

$$\rho_{ij} = \begin{pmatrix} p^+_L & 0 & 0 & \beta_L \\ 0 & p_L & \alpha_L & 0 \\ 0 & \bar{\alpha}_L & p'_L & 0 \\ \bar{\beta}_L & 0 & 0 & p^-_L \end{pmatrix}, \quad L = |i-j| \qquad (9)$$

Important case: Mixture of two aligned spin states [6,8,9]

$$\rho_{ij}(\theta) = \tfrac{1}{2}(|\theta\theta\rangle\langle\theta\theta| + |-\theta,-\theta\rangle\langle-\theta,-\theta|) \qquad (10)$$

where $|\theta\theta\rangle \equiv |\theta\rangle\otimes|\theta\rangle$ with $|\theta\rangle = \exp[i\theta s_{iy}]|0\rangle$. State (10) is separable $\forall\,\theta$: $E(A,B)=0$. But $I^B_f > 0$ for $\forall\,\theta\in(0,\pi/2)$ [5,8,9].
(10) is a particular case of (9). Represents the reduced two-qubit state for any separation L in the GS of the spin chain (8) in the immediate vicinity of the factorizing field B_s [6,8,9],where the GS is

$$|\Theta^\pm\rangle = \frac{|\Theta\rangle \pm |-\Theta\rangle}{\sqrt{2(1\pm\langle-\Theta|\Theta\rangle)}}\,, \quad |\Theta\rangle = |\theta_1\theta_2\ldots\theta_n\rangle$$

XY spin chain in a transverse field II

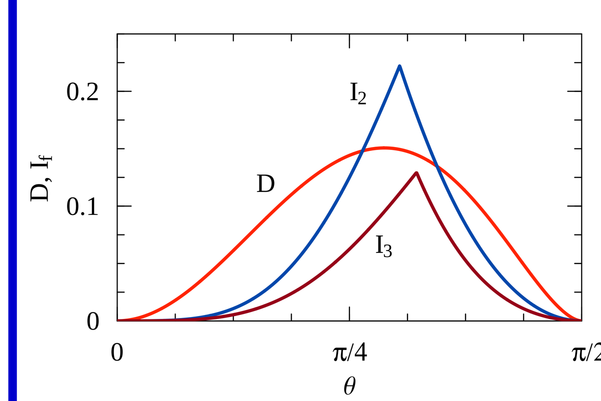


Fig. 1: Quantum correlation measures in the mixture of aligned states (10): The quantum discord D, the geometric discord I_2 and the "cubic" discord I_3, as a function of the angle θ. Normalization is such that all measures take the value 1 in a maximally entangled two-qubit state. Due to the symmetry of the state, $D = D^A = D^B$ and $I_f = I^A_f = I^B_f\;\forall\, f$ [9].

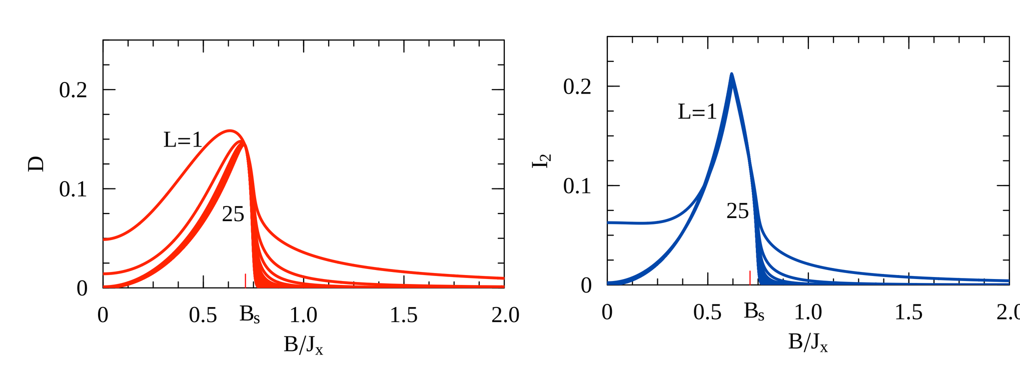


Fig. 2: The quantum discord D (left) and the geometric discord I_2 (right) between spin pairs with separation $L = 1, 2, \ldots, n/2$ in the exact ground state of a cyclic chain of $n = 50$ spins with first neighbor XY couplings and anisotropy $\chi = 0.5$. The results for all separations are simultaneously depicted. They all merge at the factorizing field B_s, where they coincide with the result for the mixture (10) with $\cos\theta = B_s/J_x = \sqrt{\chi}$.[9]

XX spin chain in a tranverse field I

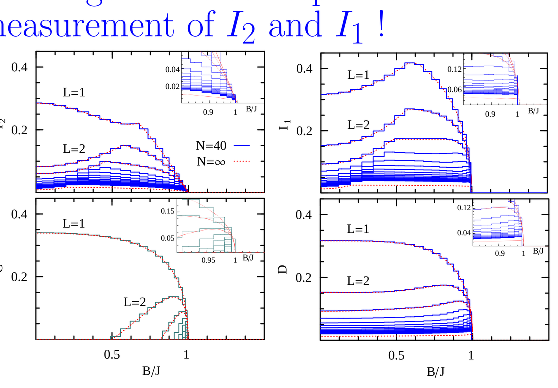
The reduced state for this model ($J^{ij}_x = J^{ij}_y = J$ in (8)) is

$$\rho_L = p^+_L|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + p^-_L|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + (p_L+\alpha_L)|\Psi_+\rangle\langle\Psi_+| + (p_L-\alpha_L)|\Psi_-\rangle\langle\Psi_-|$$

with $|\Psi_\pm\rangle$ Bell states. Finite value of D and $I_f\;\forall\,\alpha_L\neq 0$ [10].
Transition $|\Psi_+\rangle \to |\downarrow\downarrow\rangle$ in dominant eigenvalue corresponds to $\perp\to z$ transition in minimizing measurement of I_2 and I_1 !
Geometric discord:

$$I_2 = \begin{cases} I^z_2 = 4\alpha^2_L\,, & |\alpha_L| \leq \alpha^t_L \\ I^\perp_2 = 2(\alpha^2_L + {\alpha^t_L}^2)\,, & |\alpha_L| \geq \alpha^t_L \end{cases}$$

No transition in D: $D = D_\perp\;\forall\,\alpha\neq 0$.



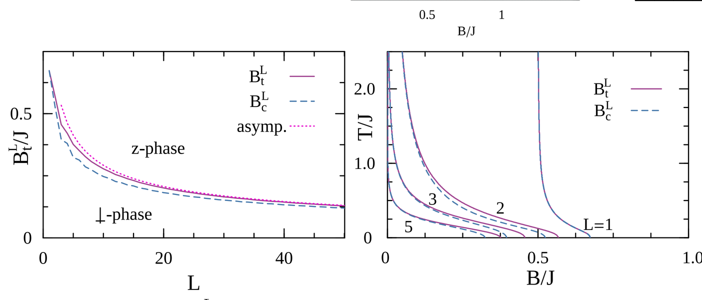
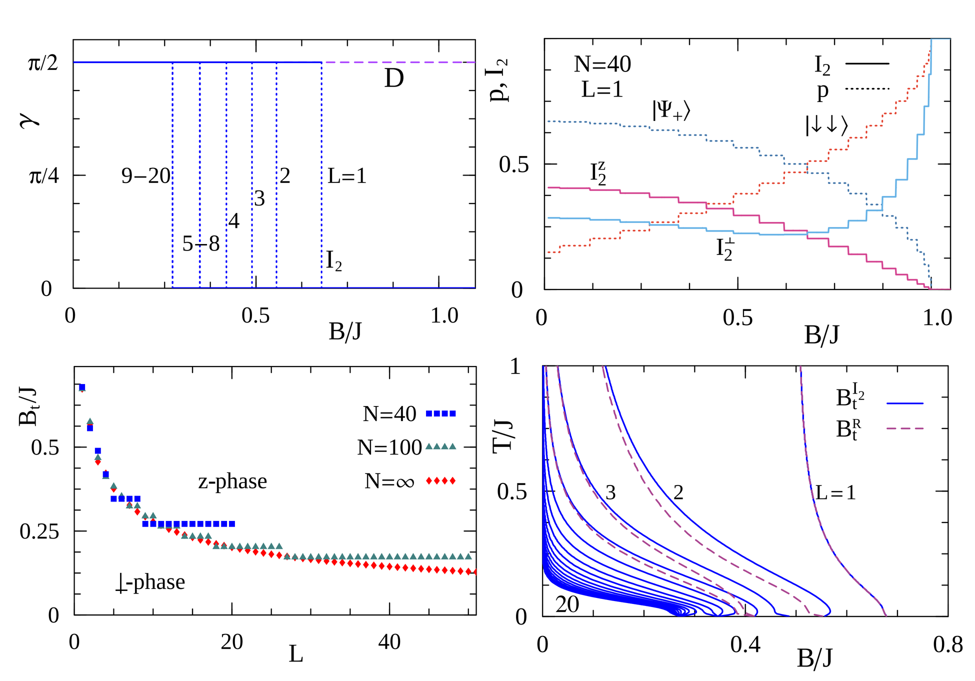


Fig. 3: Left: The $T=0$ transition field B^L_t where the measurement minimizing I_2 changes from perpendicular to parallel [10], as a function of L (solid line), together with the $T=0$ field B^L_c where the dominant eigenvector of ρ_L changes from a Bell state to an aligned state (dashed line). Both fields coincide for $L=1$ and $L\to\infty$ [10]. Dotted line indicates the asymptotic result for large L. Right: The transition fields $B^L_t(T)$ of the geometric discord for $T\neq 0$, for $L = 1,2,3$ and 5, such that $I_2 = I^\perp_2$ (I^z_2) for $B < B^L_t(T)$ ($> B^L_t(T)$). Dashed lines depict again the fields $B^L_c(T)$. For $L=1$, both fields coincide exactly $\forall\, T$, approaching $J/2$ for high T, whereas for $L\geq 2$ they merge for high T,vanishing as $(J/T)^{L-1}$ [10]

XX chain in a transverse field II



Top: Left: The minimizing angle for the geometric discord I_2 as a function of the magnetic field for spin pairs with separations $L = 1, \ldots, N/2$, in a finite chain with $N=40$ spins. Dotted lines indicate the sharp $\perp\to z$ transitions for different L. No transition occurs in the quantum discord D (dashed line), where $\gamma = \pi/2\;\forall\; B$ and L. Right: Results for the geometric discord $I^\perp_2$ and I^z_2 (solid lines) for $N=40$ and $L=1$, together with the two dominant eigenvalues of ρ_1 (dotted lines). Both cross at the same step. Bottom: Left: Exact transition fields B^L_t delimiting the $\perp$ and z phases of I_2 at $T=0$ for $N=40$, $N=100$ and the thermodynamic limit. Right: The geometric discord "phase" diagram in the finite chain of $N=40$ spins, for all separations $L = 1, \ldots, N/2$ (solid lines). The z ($\perp$) phases lie to the right (left) of these curves. Dashed lines depict the fields $B^L_c(T)$ for $L\leq 4$, below which the Bell state becomes dominant in ρ_L.[10]

Conclusions

- All measures indicate the presence of long range pairwise discord-type quantum correlations for $|B| < B_c$ in the exact GS of these chains.
- All pairwise quantum correlations reach full range at B_s, adquiring a finite non-negligible constant value independent of the pair separation or coupling range and is determined solely by the coupling anisotropy.
- Substantial differences in the minimizing local spin measurement that defines these measures: for the Quantum Discord it always prefers here a direction orthogonal to the transverse field, whereas for Information Deficit-type measures it exhibits instead a transition, from perpendicular to parallel to the field, as the latter increases, present for all pair separations and at all temperatures.
- Such behavior signatures the transition exhibited by the dominant eigenstate of the reduced state of the pair, which changes from a maximally entangled state to a separable state in the vicinity of the measurement transition.
- For contiguous pairs both transitions occur *exactly* at the same field, at all temperatures, in geometric discord. For general separations there is also exact agreement between both fields at high T, for *all measures* I_f.

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